MA4262 Measure and Integration Notes

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Contents

1. Background and Abstract Measure

In this course, we assign weightage to a certain mathematical object.

(1). **Analysis aspect:** for Measure Theory, aim to understand the volume of a set; for integration, aim to understand the mass of an object with some density function.

(2). **Probability theory aspect:** for Measure Theory, aim to understand the probability of an event; for integration, aim to understand the expectation of a random variable.

Historically, earlier attempt by Riemann who came up with the Riemann integral (define mass and volume concurrently). For simplicity, consider an arbitrary function $f:\mathbb{R}\to\mathbb{R}$. Then,

$$\int_a^b f(x)\ dx = \sum_{i=1}^{n} \Delta(x_i) f(x_i), \quad \text{where } \Delta x_i = x_i - x_{i-1}.$$

Let $P=\{x_0,x_1,\ldots,x_n\}$ denote an arbitrary partition of $[x_0,x_n]=[a,b]$. So, $[a,b]$ can be *broken down* into compact intervals $[a,x_1],\ldots,[x_{n-1},b]$. So,

$$\sum_{i=1}^{n} f(\xi_i)\Delta(x_i) = S(f,P)(\xi), \quad \text{where } \xi = (\xi_0,\ldots,\xi_n).$$

Theorem 1.1 (Riemann integrability criterion)**.** Let $P=\{x_0,\ldots,x_n\}$ be a partition and $\|P\| = \max_i |\Delta_i|$ denote its norm function. If there exists $A\in\mathbb{R}$ such that for all $\varepsilon>0$, there exists $\delta>0$ such that

$$|S(f,P)(\xi)-A|<\varepsilon \text{ holds} \quad \text{whenever} \quad \|P\|<\delta,$$

we say that f is Riemann integrable on $[a,b]$. We also let

$$\int f(x)\ dx = A \quad \text{denote} \quad \text{the Riemann integral of } f.$$

Definition 1.1 (characteristic function)**.** Say we have a region $X\subseteq\mathbb{R}^n$. Define

$$\mathrm{Vol}(X)=\int_{\mathbb{R}^n}\mathbf{1}_X(x)\ dx \quad \text{where} \quad \mathbf{1}_X:\mathbb{R}^n\to\mathbb{R} \text{ such that } x\mapsto\begin{cases}1 & \text{if } x\in X;\\ 0 & \text{if } x\notin X.\end{cases}$$

This is the idea of a characteristic function.

What is wrong with the Riemann integral? We wish to define volume and integral in a more general domain. A *nice* property of the compact interval $[a,b]$ is that we can construct a partition P of it.

Example 1.1 (coin flipping)**.** Let $\Omega=\{H,T\}^{\mathbb{N}}$ and define $A=\{(s_1,\ldots):s_1=H\}$. Goal: A has volume $1/2$.

Example 1.2. Consider

$$\mathrm{SO}_3(\mathbb{R})=\left\{\mathbf{Q}\in\mathscr{M}_{3\times3}(\mathbb{R}):\mathbf{Q}^{\mathrm{T}}\mathbf{Q}=\mathbf{Q}\mathbf{Q}^{\mathrm{T}}=\mathbf{I},\det(\mathbf{Q})=1\right\},$$

which is the special orthogonal group of 3×3 matrices. Define the set $A = \{\mathbf{Q} : \angle(\mathbf{z}, \mathbf{Qz}) \leq \pi/2\}$. We want A to have volume $1/2$. Say $\mathbf{z}$ (or rather, $\mathbf{k}$) points vertically upward (to the north pole). Think of this as deconstructing the 2-sphere $\mathbb{S}^2$ into an upper and lower half.

Example 1.3. There exist functions which are *intuitively* integrable but not Riemann integrable. For example, consider the characteristic function of $\mathbb{Q} \cap [0,1]$, defined to be

$$\mathbf{1}_{\mathbb{Q}\cap[0,1]}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \cap [0,1]; \\ 0 & \text{if } x \notin \mathbb{Q} \cap [0,1]. \end{cases}$$

Note that $\mathbb{Q}$ is countable but $[0,1]$ is uncountable. So, $\mathbb{Q} \cap [0,1]$ has *few elements* (more formally, countably many points). We want

$$\int \mathbf{1}_{\mathbb{Q}\cap[0,1]}(x)\ dx = 0.$$

However, $\mathbf{1}_{\mathbb{Q}\cap[0,1]}$ is not Riemann integrable because simply said, this function does not satisfy the Riemann integrability criterion.

A manifestation: There are natural theorems that are true but difficult to prove. These usually involve interchanging the order of summation and integral. Consider the following theorem:

Theorem 1.2 (Lebesgue's dominated convergence theorem)**.** Suppose $\{f_n\}$ is a sequence of Riemann-integrable functions on $[a,b]$ such that $f_n \to f$ pointwise for all $x \in [a,b]$. Assume that there exists some $M \in \mathbb{N}$ such that $|f_n| < M$. Then,

$$\lim_{n\to\infty} \int_a^b f_n(x)\ dx = \int_a^b f(x)\ dx.$$

2. Concrete Measure

2.1. *Total Measure and its Failure*

Example 2.1 (MA4262 AY24/25 Sem 1 Tutorial 1)**.** Prove or disprove the following statements:

(a) If $E \subseteq \mathbb{R}$ is nonempty, then $\inf E < \sup E$.

(b) There are subsets of real numbers which are both open and closed.

(c) The set $\mathbb{Q} \cap (0,1)$ is an open subset of $\mathbb{R}$.

(d) Let $F_1 \supseteq F_2 \supseteq \ldots$ be a sequence of closed subsets of $\mathbb{R}$. Then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$. □

Solution.

(a) This statement is false. Firstly, a fun fact is that if $E = \emptyset$ (of course, we restricted to $E \neq \emptyset$ though), then $\inf E = \infty$ and $\sup E = -\infty$ so $\inf E > \sup E$. Anyway, choose $x \in E$. Then, by definition, $\inf E \leq x \leq \sup E$ since $\inf E$ and $\sup E$ are lower and upper bounds for E respectively. Equality holds if E is a singleton, i.e. $E = \{x\}$ for some $x \in \mathbb{R}$, so we have disproved the statement.

(b) True, in fact, such sets are said to be clopen. For example, $\emptyset$ and $\mathbb{R}$ are both open and closed.

(c) False. Let $S = \mathbb{Q} \cap (0,1)$. Recall that a set $S \subseteq \mathbb{R}$ is said to be open if for all $x \in S$, there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subseteq S$. Let $x \in S$ be an arbitrary rational number contained in $(0,1)$. Then, no matter how small ε is, $(x - \varepsilon, x + \varepsilon)$ will contain both rational and irrational numbers since $\mathbb{Q}$ is dense in $\mathbb{R}$.

(d) The statement is true. Note that $\emptyset$ is not a subset of $\emptyset$ (even though $\emptyset$ is a closed subset of $\mathbb{R}$), so we can take each F_i to be closed intervals of $\mathbb{R}$. Say $F_1 = [a_1, b_1]$ and $F_2 = [a_2, b_2]$ such that $a_1 \leq a_2 \leq b_1 \leq b_2$. Recursively, define

$$F_n = [a_n, b_n] \quad \text{such that} \quad a_1 \leq \ldots \leq a_n \leq b_n \leq \ldots \leq b_1.$$

It is clear that $F_1 \subseteq F_2 \subseteq \ldots F_n \subseteq \ldots$ so

$$\bigcap_{n=1}^{\infty} F_n = \lim_{N \to \infty} \bigcap_{n=1}^{N} F_n = \lim_{N \to \infty} [a_N, b_N]$$

Let $c \in [a_N, b_N]$. Then, $a_N \leq c \leq b_N$ for all $N \in \mathbb{N}$. As $N \to \infty$, we see that $[a_N, b_N]$ shrinks to the singleton $\{c\}$. □

Example 2.2 (MA4262 AY24/25 Sem 1 Tutorial 1)**.** Let $X \subseteq \mathbb{R}$ be a countable set. Show that there is some $a \in \mathbb{R}$ such that $X \cap (a + X) = \emptyset$ where $a + X = \{a + x : x \in X\}$.

Solution. We have $X = \{x_1, \ldots\} = \{x_i : i \in I\}$. Let $x \in X \cap (a + X)$. Then, $x \in X$ and $x \in (a + X)$, which imply that there exist $x_i, x_j \in X$ such that $x = x_i$ and $x = a + x_j$ respectively. We can pick $a \in \mathbb{Q}' \subseteq \mathbb{R}$. Then, $a = x_i - x_j$. By definition of X, there exists $k \in I$ such that $x_k = x_i - x_j$, so $X \ni a = x_k$, which is a contradiction since $\mathbb{Q}'$ is uncountable. □

Example 2.3. $(\mathbb{Q} \cap [0,1]) \times \mathbb{R}$ is not measurable.

Definition 2.1 (power set). Let Ω be a set domain under consideration. Define $\mathscr{P}(\Omega) = \{x : x \subseteq \Omega\}$ to be the power set of Ω, i.e. the collection of subsets of Ω.

Proposition 2.1. Here are some arithmetic rules for ∞.

(1) $\infty + \infty = x + \infty = \infty + x = \infty$ for all $x \in \mathbb{R}$

(2) $(-\infty) + (-\infty) = -\infty$

(3) $\infty \cdot \infty = \infty$ and $(-\infty) \cdot \infty = -\infty$

(4) $(-\infty) \cdot \infty = -\infty$ and $(-\infty) \cdot (-\infty) = \infty$

(5) $x \cdot \infty = \infty \cdot x = \begin{cases} \infty & \text{if } x > 0; \\ -\infty & \text{if } x < 0; \\ 0 & \text{if } x = 0. \end{cases}$

The very last property on $0 \cdot \infty = \infty \cdot 0 = 0$ seems absurd but this is just by convention.

Definition 2.2 (measure/total measure). A function $\mu : \mathscr{P}(X) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is said to be a measure on Ω if the following properties are satisfied:

(a) $\mu(\emptyset) = 0$

(b) $\mu(X) \geq 0$ for all $X \in \mathscr{P}(\Omega)$

(c) μ is countably additive, i.e. if $\{X_n\}$ is a sequence of disjoint subsets of Ω, then

$$\mu\left(\bigcup_{n=0}^{\infty} X_n\right) = \sum_{n=0}^{\infty} \mu(X_n) = \lim_{N \to \infty} \sum_{n=0}^{N} \mu(X_n).$$

Axiom 2.1 (axiom of choice). Suppose $\{X_n\}$ is a family of non-empty subsets of Ω. Then, there exists (i.e. we can choose) a sequence $\{a_n\}$ of elements in Ω such that $a_n \in X_n$.

Definition 2.3 (invariant measure). For $X \subseteq \mathbb{R}$ and $a \in \mathbb{R}$, define $a + X = \{a + x : x \in X\}$. A total measure μ on $\mathbb{R}$ is invariant if

$$\mu(a + X) = \mu(X) \quad \text{for all } a \in \mathbb{R}, X \subseteq \mathbb{R}.$$

Example 2.4. $\mu(X) = 0$ for all $X \subseteq \Omega$ is a total measure.

Example 2.5. Let X be a set. Then,

$$\mu(X) = \begin{cases} |X| & \text{if } X \text{ is finite;} \\ \infty & \text{if } X \text{ is infinite.} \end{cases}$$

Example 2.6 (MA4262 AY24/25 Sem 1 Tutorial 1). If $\mu_1, \ldots, \mu_n$ are total measures on Ω and $a_1, \ldots, a_n$ are nonnegative real numbers. Show that the function λ, defined for $X \in \mathscr{P}(\Omega)$ by

$$\lambda(X) = \sum_{j=1}^{n} a_j \mu_j(X)$$

is a total measure on Ω.

Solution. We need to prove that $\lambda(\emptyset) = 0$, $\lambda(X) \geq 0$ for all $X \in \mathscr{P}(\Omega)$, and λ is countably additive.

Since μ_j is a total measure for all $1 \leq j \leq n$, then $\mu_j(\emptyset) = 0$, so the first property that $\lambda(\emptyset) = 0$ follows. Next, since $\mu_j(X) \geq 0$ and $a_1, \ldots, a_n \geq 0$, which implies the second property that $\lambda(X) \geq 0$. To prove the third property, let $X_1, \ldots$ be a sequence of disjoint subsets of Ω. Since μ is a measure on Ω, then

$$\mu\left(\bigcup_{n=0}^{\infty} X_n\right) = \sum_{n=0}^{\infty} \mu(X_n).$$

So,

$$\begin{aligned}
\lambda\left(\bigcup_{i=0}^{\infty} X_i\right) &= \sum_{j=1}^{n} a_j \mu_j \left(\bigcup_{i=0}^{\infty} X_i\right) \\
&= \sum_{j=1}^{n} a_j \sum_{i=0}^{\infty} \mu(X_i) \quad \text{since } \mu \text{ is a measure on } \Omega \\
&= \sum_{i=0}^{\infty} \sum_{j=1}^{n} a_j \mu(X_i) \quad \text{since } a_1, \ldots, a_n \text{ are constants} \\
&= \sum_{i=0}^{\infty} \lambda(X_i)
\end{aligned}$$

which shows that λ satisfies the third property. □

Example 2.7. Let Ω be a countably infinite set, which you can take as $\mathbb{N} = \{0, 1, \ldots\}$, and let $\mathscr{P}(\Omega)$ be the family of all subsets of Ω. Define

$$\mu : \mathscr{P}(\Omega) \to \mathbb{R} \cup \{\pm\infty\} \quad \text{such that} \quad \mu(X) = \begin{cases} 0 & \text{if } X \text{ is finite;} \\ \infty & \text{if } X \text{ is infinite.} \end{cases}$$

Prove or disprove whether μ is a total measure on Ω.

Solution. By definition, $\mu(X)$ is either 0 or ∞, both of which are non-negative. Furthermore, the empty set is finite, so $\mu(\emptyset) = 0$. However, μ is not countably subadditive. To see why, let X_i be disjoint subsets of $\mathscr{P}(\Omega)$, i.e. $X_i = \{i\}$ for $i \in \mathbb{N}$. Then,

$$\infty = \mu\left(\bigcup_{i=0}^{\infty} X_i\right) = \sum_{i=0}^{\infty} \mu(X_i) = 0$$

where we assumed the countable subadditivty of μ, which leads to a contradiction. □

Example 2.8. Let Ω be an uncountable set, which you can take as the set $\mathbb{R}$ of real numbers, for example, and let $\mathscr{P}(\Omega)$ be the family of all subsets of Ω. Define

$$\mu : \mathscr{P}(\Omega) \to \mathbb{R} \cup \{\pm\infty\} \quad \text{such that} \quad \mu(X) = \begin{cases} 0 & \text{if } X \text{ is countable;} \\ \infty & \text{if } X \text{ is uncountable.} \end{cases}$$

Prove or disprove whether μ is a total measure on Ω.

Solution. Clearly, μ satisfies the non-negativity and null empty set conditions. We now verify that μ is countably additive. To do so, we shall consider two cases — firstly if all X_i are countable, and secondly if at least one of the X_i's is uncountable.

If all X_i are countable, then

$$0 = \mu\left(\bigcup_{i=1}^{\infty} X_i\right) = \sum_{i=1}^{\infty} \mu(X_i) = 0.$$

Here, we used the fact that the countable union of countable sets is countable (see theorem below). On the other hand, if at least one of the X_i's is uncountable, then we have

$$\bigcup_{i=1}^{\infty} X_i \quad \text{being an uncountable set}$$

since the union of a countable set and an uncountable set is uncountable. Hence,

$$\infty = \mu\left(\bigcup_{i=1}^{\infty} X_i\right) = \sum_{i=1}^{\infty} \mu(X_i) = \infty.$$

It follows that μ satisfies the countably additive property, so indeed, μ is a total measure on Ω. □

In the previous example, we used the following fact which was proven in MA2101S AY24/25 Sem 1 Tutorial 1 Question 2(**a**):

Theorem 2.1 (countable union of countable sets is countable)**.** The countable union of countable sets is countable; i.e., if I is countable, and $\{S_i : i \in I\}$ is a collection of countable sets, then $\bigcup_{i \in I} S_i$ is countable.

Proof. Note that $S_1, S_2, \ldots$ are countable. Assume that these sets do not have any elements in common, otherwise consider $S_1' = S_1$, $S_2' = S_2 \backslash S_1$, $S_3' = S_3 \backslash (S_1 \cup S_2)$, and so on. In general, for $n \geq 2$, we have

$$S_n' = S_n \backslash \bigcup_{i=1}^{n-1} S_i$$

and we note that S_i' is countable for all $i \in I$ since any subset of a countable set is countable. We enumerate the elements of $S_1', S_2', \ldots$ in the following table:

$$\begin{array}{cccc} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

Here, a_{ij} is the j^{th} element of S_i. So, $\phi : S \to \mathbb{N} \times \mathbb{N}$, where $a_{ij} = (i, j)$ is injective. It is a known fact that $\psi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is injective, so $\psi \circ \phi : S \to \mathbb{N}$ is injective. It follows that S is countable. □

Example 2.9. Suppose μ is an invariant total measure on $\mathbb{R}$ such that $\mu([0,1]) = 1$. Show that $\mu([a,b]) = b - a$ for $a, b \in \mathbb{R}$ such that $a < b$ and $\mu(\mathbb{R}) = \infty$.

Solution. Let $X \subseteq \mathbb{R}$ and $a \in \mathbb{R}$. Recall that a total measure μ is said to be invariant if $\mu(a+X) = \mu(X)$. Suppose $X = [0,1]$ and $a \in \mathbb{R}$. Then,

$$a+X = \{a+x : x \in [0,1]\} = \{y : y \in [a, 1+a]\}.$$

So, $\mu([a,1+a]) = \mu([0,1]) = 1$. We can repeat this process to obtain the following result for any $n \in \mathbb{R}^+$:

$$\begin{aligned} \mu([a,1+a]) + \mu([1+a,2+a]) + \ldots + \mu([n-1+a,n+a]) &= n \\ \mu([a,n+a]) &= n \end{aligned}$$

Set $b = n+a$, so $n = b-a$, which implies $\mu([a,b]) = b-a$.

For the second part, we can think of

$$\mathbb{R} = \lim_{R \to \infty} [-R,R].$$

So, we shall set $b = R$ and $a = -R$, which we obtain $\mu([-R,R]) = 2R$. By countable additivity,

$$\mu\left(\bigcup_{R=1}^{\infty} [-R,R]\right) = \sum_{R=1}^{\infty} \mu[-R,R] \quad \text{which implies} \quad \mu(\mathbb{R}) = \sum_{R=1}^{\infty} 2R = \infty$$

so the second result follows. □

Theorem 2.2 (Vitali). There does not exist a total measure which is invariant on $\mathbb{R}$ such that $\mu([0,1]) = 1$.

Intuitively, why is this wrong? If μ is translation invariant and $\mu([0,1]) = 1$, then for any $n \in \mathbb{Z}$, $\mu([n,n+1])$ should also be 1. By countable additivity, $\mu(\mathbb{Z}_{\geq 0}) = \infty$ because it would be the sum of infinitely many 1's.

Proof. Let $\sim$ be the relation on $\mathbb{R}$ given by

$$a \sim b \quad \text{if and only if} \quad a-b \in \mathbb{Q}.$$

In fact, $\sim$ is an equivalence relation. Now, the interval $[0,1] \subseteq \mathbb{R}$ can be decomposed as

$$[0,1] =_{i \in I} D_i \quad \text{where } D_i = [0,1] \cap E_i \text{ with } E_i \text{ an equivalence class of } \sim.$$

We say that D_i is a Vitali set. Note that $[0,1]$ is uncountable. Since E_i is an equivalence class of $\sim$, then D_i is countable (recall the quotient $\mathbb{R}/\mathbb{Q}$). Since $[0,1]$ is the disjoint union of uncountable many countable sets, it is also uncountable. This is perfectly fine (have not invoked axiom of choice). Now, we use the axiom of choice to choose $i \in I$ an element $d_i \in D_i$ and set $X = \{d_i : i \in I\}$ and $0 \in X$.

How do we reach a contradiction? Well, we can do so if we can prove that there exists a total measure

μ such that $\mu(X) = 0$ and $\mu(X) > 0$. We claim that

$$[0,1] \subseteq \bigcup_{a \in [-1,1] \cap \mathbb{Q}} (a+X) \subseteq [-1,2].$$

To see why, pick $b \in [0,1]$. Then, $b \sim b'$. So, $a = b - b' \in [-1,1]$, which implies $b \in a+X$ and the first inclusion follows. The second inclusion follows from the fact that $X \subseteq [0,1]$.

- **Case 1:** Suppose $\mu(X) = 0$. Then, $\mu(a+X) = 0$ for any $a \in [-1,1] \cap \mathbb{Q}$ by invariance of total measure. So,

$$\mu\left(\bigcup_{a \in [-1,1] \cap \mathbb{Q}} (a+X)\right) = 0 \quad \text{since } (a+X) \cap \left(a' \cap X\right) = \emptyset \text{ for } a \neq a'.$$

  Indeed, if $b \in (a+X) \cap (a'+X)$, then $b-a, b-a' \in X$. Since $b-a \sim b-a'$, then $b-a = b-a'$, implying that $a = a'$. Thus,

$$\begin{aligned} \mu([0,1]) &= \mu\left(\bigcup_{a \in [-1,1] \cap \mathbb{Q}} (a+X)\right) - \mu\left(\left(\bigcup_{a \in [-1,1] \cap \mathbb{Q}} (a+X)\right) \setminus [0,1]\right) \\ &= 0 - 0 \quad \text{where second quantity must be 0 by non-negativity of } \mu \end{aligned}$$

  which implies $\mu([0,1]) = 0$, but this is clearly a contradiction.
- **Case 2:** Suppose $\mu(X) = c > 0$. In fact, note that $c \leq 1$. Again, by invariance, $\mu(a+X) = c$. Then,

$$\mu\left(\bigcup_{a \in [-1,1] \cap \mathbb{Q}} (a+X)\right) = \infty \quad \text{since the LHS is the countable union of countable sets.}$$

  However, the union is a subset of $[-1,2]$, which implies $\infty \leq 3$, a contradiction again.

□

Remark 2.1. 2 ways to move forward:

- **Axiom of choice is wrong:** Refer to the works of Robert Solovay. Use Set Theory to construct a universe where the axiom of choice is wrong to obtain a total measure. Some portion of the axiom of choice is still true enough for most purposes.
- **Notion of total measure is wrong:** We only need μ well-defined on sets that are nice enough

Example 2.10. Suppose V is a Vitali set. Show that

$$\text{there exists a sequence of real numbers } a_n \quad \text{such that} \quad \mathbb{R} = \bigcup_n (a_n + V).$$

Solution. We shall select one representative from each equivalence class of $\mathbb{R}/\mathbb{Q}$, i.e. for each $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $x - y \in \mathbb{Q}$. Note that $V \cap (V+r) = \emptyset$ for any $r \in \mathbb{Q}$ but we can use these translates to cover $\mathbb{R}$. Let $a_n = q_n$ denote the sequence of rationals, which is countable, so that the result follows. □

2.2. *σ-Algebras and Measures*

Definition 2.4 (σ-algebra)**.** A sub-collection $\mathscr{A}$ of $P(\Omega)$ is a σ-algebra if one of the following conditions are satisfied:

(i) $\emptyset, \Omega \in \mathscr{A}$

(ii) If $X, Y \in \mathscr{A}$, then $X \backslash Y \in \mathscr{A}$

(iii) If $\{X_n\}$ is a sequence of disjoint members of $\mathscr{A}$, then

$$\bigcup_{n=1}^{\infty} X_n \in \mathscr{A}.$$

Remark 2.2. Note that if condition **(iii)** in Definition 2.4 of a σ-algebra is replaced by saying that if $X_1, X_2 \in \mathscr{A}$ are disjoint, then $X_1 \cup X_2 \in \mathscr{A}$, then we obtain the definition of a **Boolean algebra**.

Example 2.11 (trivial examples of σ-algebras)**.** $\mathscr{A} = \{\emptyset, \Omega\}$ and $\mathscr{A} = \mathscr{P}(\Omega)$ are obvious σ-algebras.

Example 2.12 (intersection of σ-algebras also a σ-algebra)**.** If $\mathscr{A}_1, \mathscr{A}_2$ are σ-algebras, then $\mathscr{A}_1 \cap \mathscr{A}_2$ is also a σ-algebra. More generally, if $\{A_i\}_{i \in I}$ is a family of σ-algebras, then $\bigcap_{i \in I} A_i$ is also a σ-algebra.

Example 2.13. The following example is relatively common. Suppose $\{X_n\}$ is a sequence of members of $\mathscr{A}$ which are not necessarily disjoint. Then, we enumerate the members $X_1, X_2, X_3, \ldots$ and we shall also define Y_i as follows:

$$Y_1 = X_1 \quad \text{and} \quad Y_n = X_n \backslash \bigcup_{i=1}^{n-1} X_i$$

For instance, $Y_2 = X_2 \backslash X_1$, $Y_3 = X_3 \backslash (X_1 \cup X_2)$ and so on. Then, it is again clear that $\mathscr{A}$ is a σ-algebra.

Example 2.14 (smallest σ-algebra is Borel)**.** Suppose $\mathscr{C} \subseteq \mathscr{P}(\Omega)$. Then, there exists a smallest σ-algebra containing $\mathscr{C}$ known as the σ-algebra generated by $\mathscr{C}$ (will visit Borel σ-algebra in due course).

Example 2.15 (Axler p. 38 Question 1)**.** Show that

$$S = \left\{ \bigcup_{n \in K} (n, n+1] : K \subseteq \mathbb{Z} \right\} \quad \text{is a } \sigma\text{-algebra on } \mathbb{R}.$$

Solution. Suppose $K = \emptyset$, then $\emptyset \in S$. Let

$$A \in S \quad \text{so} \quad A = \bigcup_{n \in K} (n, n+1] \text{ for some } K \subseteq \mathbb{Z}.$$

Then,

$$\mathbb{R} \backslash A = \bigcup_{n \in \mathbb{Z} \backslash K} (n, n+1] \quad \text{so the complement of } A \text{ in } \mathbb{R} \text{ is also the union of half-open intervals.}$$

Lastly, consider $A_1, \ldots$ which are disjoint such that

$$A_i = \bigcup_{n \in K_i} (n, n+1] \quad \text{for some} \quad K_i \subseteq \mathbb{Z}, \text{ where } i \in \mathbb{N}.$$

Then,

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \bigcup_{n \in K_i} (n, n+1] \in S$$

so indeed, S is a σ-algebra on $\mathbb{R}$. □

Lemma 2.1. Let $\mathscr{A} \subseteq \mathscr{P}(\mathbb{R}^n)$. The following are equivalent:

(a) $\mathscr{A}$ is the algebra generated by open boxes, which are sets of the form $I_1 \times I_2 \times \ldots \times I_n$, where each I_k is an open interval, i.e. either $(-\infty, a_k), (a_k, b_k), (b_k, \infty), (-\infty, \infty)$.

(b) Similar to **(b)**, $\mathscr{A}$ is the algebra generated by closed boxes, which are sets of the form $I_1 \times I_2 \times \ldots \times I_n$, where each I_k is a closed interval, i.e. either $(-\infty, a_k], [a_k, b_k], [b_k, \infty), (-\infty, \infty)$.

(c) $\mathscr{A}$ consists of sets which are disjoint unions of boxes, i.e. sets of the form $I_1 \times \ldots \times I_n$, with examples of I_k being $(-\infty, a_k), (-\infty, a_k], (a_k, b_k], \ldots$ and the list goes on.

Lemma 2.2. Let $\mathscr{A} \in \mathscr{P}(\mathbb{R}^n)$. The following are equivalent:

(a) $\mathscr{A}$ is the σ-algebra generated by open boxes

(b) $\mathscr{A}$ is the σ-algebra generated by closed boxes

(c) $\mathscr{A}$ is the σ-algebra generated by bounded open boxes, i.e.

$$\prod_{k=1}^{n} I_k \quad \text{with} \quad I_k = [a_k, b_k].$$

(d) $\mathscr{A}$ is the σ-algebra generated by closed and bounded (in fact, compact by the Heine-Borel theorem) intervals $I_k = [a_k, b_k]$

(e) $\mathscr{A}$ is generated by open sets

(f) $\mathscr{A}$ is generated by closed sets

Definition 2.5 (Minkowski sum)**.** Consider the operation + for sets in $\mathbb{R}^n$. If $A, B \subseteq \mathbb{R}^n$, define the Minkowski sum of A and B as follows:

$$A + B = \{(a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) : (a_1, \ldots, a_n) \in A \text{ and } (b_1, \ldots, b_n) \in B\}$$

Example 2.16 (MA4262 AY24/25 Sem 1 Tutorial 2)**.** Consider the operation + for sets in $\mathbb{R}^n$. If $A, B \subseteq \mathbb{R}^n$, then set

$$A + B = \{(a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) : (a_1, \ldots, a_n) \in A \text{ and } (b_1, \ldots, b_n) \in B\}.$$

As mentioned, this is known as the Minkowski sum. For the following, true or false? Justify your answer.

(a) If A and B are open, then $A+B$ is open.
(b) If A and B are compact, then $A+B$ is compact.
(c) If A and B are closed, then $A+B$ is Borel.
(d) If A and B are closed, then $A+B$ is closed.

Solution.

(a) True, since if A and B are open sets, we consider a point $p \in A+B$, so $p = a+b$ for some $a \in A$ and $b \in B$. Since A is open, there exists an ε_1-neighbourhood centred at a, denoted by $U_{\varepsilon_1}(a)$; define $U_{\varepsilon_2}(b)$ similarly. So, $U_{\varepsilon_1} + U_{\varepsilon_2}$ is an open neighbourhood of p contained in $A+B$.

(b) True. Since A and B are compact, by sequential compactness, we have $a_k + b_k \to y$ for some sequences $a_k \in A, b_k \in B$ for all $k \in \mathbb{N}$. So, there exists a subsequence $a_{k_i} \to a$ for some $a \in A$; the same claim can be made for b. So $a_{k_i} + b_{k_i} \to a+b = y$.

Alternatively, consider $A \times B \subseteq \mathbb{R}^{2n}$ which is compact (overkill by Tychonoff's theorem). Then, consider $A \times B \to \mathbb{R}^n$ via $(a,b) \mapsto a+b$ which is continuous. Since the continuous image of a compact set is compact, the result follows.

(c) True, use a similar argument made in the alternative solution to **(b)**.

(d) False. Let

$$A = \{n : n = 1,2,\ldots\} \quad \text{and} \quad B = \left\{-n + \frac{1}{n} : n = 2,3,\ldots\right\} \quad \text{be closed subsets of } \mathbb{R}$$

We first show that they are closed sets. Note that 0 is a limit point of A, whereas B is a discrete set of points (just like A) and it contains all the limit points. However,

$$\begin{aligned} A+B &= \{a+b : a \in A \text{ and } b \in B\} \\ &= \left\{a - b + \frac{1}{b} : a = 1,2,\ldots \text{ and } b = 2,3,\ldots\right\} \end{aligned}$$

By considering the case when $a = b$, we note that 0 is a limit point of $A+B$. However, $0 \notin A+B$. □

Definition 2.6 (Borel set)**.** The smallest σ-algebra on $\mathbb{R}^n$ containing all open subsets of $\mathbb{R}^n$ is called the collection of Borel subsets of $\mathbb{R}^n$. An element of this σ-algebra is called a Borel set. The collection of all Borel sets on $\mathbb{R}^n$ forms a σ-algebra, known as the Borel algebra, denoted by $\mathscr{B}(\mathbb{R}^n)$. If $X \in \mathscr{B}(\mathbb{R}^n)$, then X is Borel.

Remark 2.3. One can define Borel algebras for an arbitrary topological space X.

Remark 2.4. The set of Borel algebras is quite robust. One can also use it for other sets of generators (i.e. boxes that are semi-open).

Example 2.17 (MA4262 AY24/25 Sem 1 Tutorial 2)**.** Show that the Borel σ-algebra $\mathscr{B}(\mathbb{R})$ is generated by the collection of all semi-open intervals $(a,b] = \{x \in \mathbb{R} \mid a < x \leq b\}$.

Solution. We first need to prove that $\mathscr{B}(\mathbb{R})$ contains all semi-open intervals, then prove that the smallest σ-algebra that contains all semi-open intervals $(a,b]$ is $\mathscr{B}(\mathbb{R})$. To prove the first assertion, we note that

$$(a,b]=\bigcap_{n=1}^{\infty}\left(a+\frac{1}{n},b\right).$$

Since $(a+1/n,b)$ is an open interval and the countable intersection of open sets is in $\mathscr{B}(\mathbb{R})$, we see that $(a,b]\in\mathscr{B}(\mathbb{R})$. As for the second assertion, we let $\mathscr{A}$ be the σ-algebra generated by the collection of half-open intervals of the form $(a,b]$. So, $\mathscr{A}$ is the smallest σ-algebra that contains all sets of the form $(a,b]$. Observe that

$$(a,b)=\bigcup_{n=1}^{\infty}\left(a+\frac{1}{n},b\right]$$

so $\mathscr{A}$ also contains all open intervals of the form (a,b), which implies $\mathscr{B}(\mathbb{R})\subseteq\mathscr{A}$. In fact, $\mathscr{B}(\mathbb{R})=\mathscr{A}$. □

Example 2.18 (MA4262 AY24/25 Sem 1 Tutorial 2)**.** Let (X_n) be a sequence of Borel sets. Suppose A,B are subsets of $\mathbb{R}$, such that A consists of all $x\in\mathbb{R}$ which belong to infinitely many of the sets X_n, and B consists of all $x\in\mathbb{R}$ which belong to all but a finite number of the sets X_n. Show that both A and B are also Borel sets.

Hint: Show that

$$A=\bigcap_{m=1}^{\infty}\left(\bigcup_{n=m}^{\infty}X_n\right),\quad B=\bigcup_{m=1}^{\infty}\left(\bigcap_{n=m}^{\infty}X_n\right).$$

Note that if we replace 'Borel sets' and '$\mathbb{R}$' by ' σ-algebra $\mathscr{A}$ on Ω' and 'Ω' respectively, the analogous result still holds.)

The set A in the above question is called the outer limit or the limit superior of (X_n); the set B in the above question is called the inner limit or the limit inferior of (X_n). We write the limit superior A and limit inferior B as

$$\limsup_{n\to\infty}X_n\quad\text{and}\quad\liminf_{n\to\infty}X_n\quad\text{respectively.}$$

Solution. For A, we have

$$x\in A\quad\text{if and only if}\quad\forall m\in\mathbb{Z}^+,\exists n\geq m\text{ such that }x\in X_n.$$

Similarly, for B, we have

$$x\in B\quad\text{if and only if}\quad m\in\mathbb{Z}^+\text{ such that }\forall n\geq m,x\in X_n.$$

We have established the hint. To deduce the main result, we only prove for A (B can be derived similarly). Note that X_n is Borel so we take the countable union of Borel sets, which implies it is also Borel. Since the intersection of countably infinite Borel sets is also Borel, the result follows. □

Example 2.19 (MA4262 AY24/25 Sem 1 Tutorial 2)**.** If (X_n) is a sequence of subsets of a set $\Omega = \mathbb{R}$ which is monotone increasing (that is, $X_1 \subseteq X_2 \subseteq \dots$), show that

$$\limsup X_n = \bigcup_{n=1}^{\infty} X_n = \liminf X_n.$$

Solution. We know that

$$\limsup X_n = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} X_n \quad \text{and} \quad \liminf X_n = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} X_n.$$

Let $x \in \limsup X_n$. Then,

$$\text{for all } m \geq 1 \text{ there exists } n \in \mathbb{N} \text{ such that for all } n \geq m \text{ we have } x \in X_n.$$

So, $x \in X_n$ for infinitely many $n \in \mathbb{Z}^+$ so it follows that $x \in X_1 \cup \dots$. To show the reverse inclusion, let $x \in X_1 \cup \dots$, then there exists $n \geq m$ such that $x \in X_n$, i.e. $x \in X_m \subseteq X_{m+1} \subseteq \dots$. Taking the intersection over all $m \in \mathbb{Z}^+$, the result follows. The proof that the lim inf is equal to the union of the X_n's is the same. □

Example 2.20. Prove that there exist $(X_n) \subseteq \Omega = \mathbb{R}$ such that $\liminf X_n = \emptyset$ and $\limsup X_n = \Omega$.

Solution. Consider

$$X_n = \begin{cases} \mathbb{R} \setminus [n, n+1] & \text{if } n \text{ is odd;} \\ \mathbb{R} & \text{if } n \text{ is even.} \end{cases}$$

We shall verify that $\liminf X_n = \emptyset$ and $\limsup X_n = \Omega$. So,

$$\begin{aligned} \limsup X_n &= \bigcap_{m=1}^{\infty} \bigcup_{n \geq m} X_n \\ &= \bigcap_{m=1}^{\infty} (\mathbb{R} \setminus [m, m+1]) \cup \mathbb{R} \cup (\mathbb{R} \setminus [m+2, m+3]) \cup \dots \quad \text{(note that there is a similar case)} \\ &= \mathbb{R} \end{aligned}$$

Also,

$$\begin{aligned} \liminf X_n &= \bigcup_{m=1}^{\infty} \bigcap_{n \geq m} X_n \\ &= \bigcup_{m=1}^{\infty} (\mathbb{R} \setminus [m, m+1]) \cap \mathbb{R} \cap (\mathbb{R} \setminus [m+2, m+3]) \cap \dots \quad \text{(note that there is a similar case)} \\ &= \emptyset \end{aligned}$$

□

Example 2.21 (MA4262 AY24/25 Sem 1 Tutorial 2)**.**

(a) Let $f:\mathbb{R}\to\mathbb{R}$ be any function. For any $a\in\mathbb{R}$, let

$$\omega_f(a)=\lim_{\delta\to 0}\sup\left\{\left|f\left(x'\right)-f\left(x''\right)\right|:a-\delta<x',x''<a+\delta\right\}.$$

The function ω_f is called the oscillation or amplitude function for f. Show that for any $t\in\mathbb{R}$, the set

$$H=\left\{x\in\mathbb{R}:\omega_f(x)<t\right\}\quad\text{is open.}$$

(b) Show that f is continuous at a iff $\omega_f(a)=0$.

(c) Show that the set

$$C=\{x\in\mathbb{R}:f\text{ is continuous at }x\}$$

is an intersection of countably many open subsets of $\mathbb{R}$. Deduce that C is Borel.

Solution.

(a) By definition, for all $\varepsilon>0$, there exists $\delta>0$ such that for all $x',x''\in(a-\delta,a+\delta)$, we have $|f(x')-f(x'')|<t-\varepsilon$. Here, $t>\varepsilon>0$. Now, consider a point $b\in(a-\delta,a+\delta)$, then $|f(x')-f(x'')|<t-\varepsilon<t$.

So, for every point $a\in H$, there exists a neighbourhood entirely contained in H.

(b) We first prove the forward direction. Say f is continuous at a. Then, for all $\varepsilon>0$, there exists $\delta>0$ such that $|f(x)-f(a)|<\varepsilon$. Setting $x'=x$ and $a=x''$, we have $|f(x)-f(a)|<\varepsilon$, where $a-\delta<x,a<a+\delta$. Note that

$$a-\delta<a<a+\delta\quad\text{is equivalent to }-\delta<0<\delta.$$

However, since ε can be made arbitrarily small, it follows that $w_f(a)=0$.

For the reverse direction, say $w_f(a)=0$, then

$$\lim_{\delta\to 0}\sup\left\{\left|f\left(x'\right)-f\left(x''\right)\right|:a-\delta<x',x''<a+\delta\right\}=0.$$

Then, for all $\varepsilon>0$, there exists $\delta>0$ such that whenever $x',x''\in(a-\delta,a+\delta)$, we have $|f(x')-f(x'')|<\varepsilon$. Again, we can take ε to be arbitrarily small which implies f is continuous at a.

(c) In **(b)**, f is continuous at x if and only if $\omega_f(x)=0$. Consider the sequence of sets

$$C_n=\left\{x\in\mathbb{R}:\omega_f(x)<\frac{1}{n}\right\},\quad\text{which implies}C=\bigcap_{n=1}^{\infty}C_n,$$

where each C_n is open by **(a)**. Since C is an intersection of countably many open subsets of $\mathbb{R}$, it follows that C is open, and hence Borel. □

Example 2.22 (MA4262 AY14/15 Sem 1 Tutorial 7)**.** Suppose $f:\mathbb{R}\to\overline{\mathbb{R}}$ is an extended real-valued function ($\overline{\mathbb{R}}$ means we can take both positive and negative infinity) such that

$$\left\{x\in\mathbb{R}:f(x)>k>\frac{5}{n}\right\}\quad\text{is a Borel set}\quad\text{for all }k\in\mathbb{Z},n\in\mathbb{N}.$$

Prove that f is Borel measurable.

Solution. We need to prove that for every $\alpha \in \mathbb{R}$, the set $E = \{x \in \mathbb{R} : f(x) > \alpha\}$ is a Borel set, i.e. suffices to write E as the countable union of Borel sets, which consequently implies that it is also Borel.

For each $n \in \mathbb{N}$, let k_n be the minimal integer such that $k_n/5^n > \alpha$. Then, $(k_n+1)/5^n \leq \alpha$ so

$$\lim_{n\to\infty} \left|\alpha - \frac{k_n}{5^n}\right| = 0.$$

We define

$$E = \bigcup_{n=1}^{\infty} \left\{x \in \mathbb{R} : f(x) > \frac{k_n}{5^n}\right\}$$

and the result follows. □

Remark 2.5. $\mathbb{Q} \cap [0,1] \in \mathscr{B}(\mathbb{R})$ but $\mathbb{Q} \cap [0,1]$ is not Riemann measurable, i.e. $\mathbf{1}_{\mathbb{Q}\cap[0,1]}$ is not Riemann integrable.

Definition 2.7 (measure and pre-measure)**.** A measure μ is an extended real-valued function defined on a σ-algebra $\mathscr{A}$ satisfying

(a) $\mu(\emptyset) = 0$

(b) $\mu(X) \geq 0$ for all $X \in \mathscr{A}$

(c) **countable additivity:** if $\{X_n\}$ is a sequence of disjoint sets in $\mathscr{A}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} X_n\right) = \sum_{n=1}^{\infty} \mu(X_n).$$

A pre-measure μ is an extended real-valued function on an algebra $\mathscr{A}$ satisfying **(a)**, **(b)** and if $\{X_n\}$ is a sequence of disjoint sets in $\mathscr{A}$ such that

$$\bigcup_{n=1}^{\infty} X_n \in \mathscr{A}, \quad \text{then} \quad \mu\left(\bigcup_{n=1}^{\infty} X_n\right) = \sum_{n=1}^{\infty} \mu(X_n).$$

Definition 2.8 (measure space)**.** A triple $(\Omega, \mathscr{A}, \mu)$ on $\mathscr{A}$, where is a σ-algebra on Ω, μ is a measure on $\mathscr{A}$, is said to be a measure space. If X is in $\mathscr{A}$, we sometimes say that X is μ-measurable. We also call $(\Omega, \mathscr{A})$ a measurable set.

Example 2.23. Any total measure is a measure on $\mathscr{P}(\Omega)$.

Example 2.24. On the algebra, one can define a pre-measure by setting

$$\mu(I_1 \times \ldots \times I_n) = \ell(I_1) \cdot \ldots \cdot \ell(I_n)$$

and

$$\ell(I_k) = \begin{cases} b_k - a_k & \text{if } I_k = (a_k, b_k), [a_k, b_k], (a_k, b_k], [b_k, a_k); \\ \infty & \text{otherwise.} \end{cases}$$

This can be extended to a unique measure on $\mathscr{B}(\mathbb{R}^n)$ but this requires Carathéodory's extension theorem.

Lemma 2.3. Let $(\Omega, \mathscr{A}, \mu)$ be a measure space. Then,

$$X, Y \in \mathscr{A}, X \subseteq Y \implies \mu(X) \le \mu(Y).$$

Moreover, if $\mu(X) < \infty$, then

$$\mu(Y \backslash X) = \mu(Y) - \mu(X).$$

Proof. Since $X \subseteq Y$, then $Y = (Y \backslash X) \cup X$, so $\mu(Y) = \mu(Y \backslash X) + \mu(X)$. □

Lemma 2.4. Suppose $\{X_n\}$ is an increasing sequence. Then,

$$\mu\left(\bigcup X_n\right) = \lim_{n\to\infty} \mu(X_n).$$

Proof. We assume that $X_0 = \emptyset$. Define

$$\begin{aligned}\bigcup_{n=1}^{\infty} X_n &= \bigcup_{m=1}^{\infty} (X_m \backslash X_{m-1}) \\ &= \lim_{N\to\infty} \bigcup_{m=1}^{N} (\mu(X_m) - \mu(X_{m-1})) \quad \text{by property of disjoint union} \\ &= \lim_{N\to\infty} \mu(X_N)\end{aligned}$$

Since N is a dummy variable, we can replace it when n and the result follows. □

2.3. *Carathéodory's extension theorem*

Definition 2.9 (finite measure and σ-finite measure)**.** Suppose $(\Omega, \mathscr{A}, \mu)$ is a triple such that $\mathscr{A}$ is a σ-algebra and μ is a pre-measure defined on the members of $\mathscr{A}$. We say that

(a) μ is finite if $\mu(\Omega) < \infty$;

(b) μ is σ-finite if $\Omega = \bigcup_n \Omega_n$ with each $\Omega_n \in \mathscr{A}$ and $\mu(\Omega_n) < \infty$

Example 2.25. Give an example of a measure space $(X, \mathscr{S}, \mu)$ such that

$$\{\mu(E) : E \in \mathscr{S}\} = [0,1] \cup [3,\infty].$$

Solution. Let $X = [0,1] \cup [3,\infty]$ and $\mathscr{S}$ to be the set of all measurable subsets of X. We have

$$\mu(E) = \begin{cases} x \in [0,1] & \text{if } E \subseteq [0,1]; \\ y \in [3,\infty] & \text{if } E \subseteq [3,\infty] \\ x+y, \text{ where } x \in [0,1] \text{ and } y \in [3,\infty] & \text{if } E \text{ is the disjoint union of } A \subseteq [0,1] \text{ and } B \subseteq [3,\infty]. \end{cases}$$

Note that for the third case, $\mu(E) = \mu(A) + \mu(B) = x + y$. □

Theorem 2.3 (Carathéodory's extension theorem)**.** Suppose $(\Omega, \mathscr{A}, \mu)$ is a triple where $\mathscr{A}$ is an algebra and μ is a pre-measure defined on members of $\mathscr{A}$. Then, μ can be extended to a measure μ' on the σ-algebra generated by $\mathscr{A}$. Moreover,

$$\text{if} \quad \mu \text{ is } \sigma\text{-finite} \quad \text{then} \quad \mu' \text{ is the unique such extension.}$$

We will briefly discuss the proof of Carathéodory's extension theorem in just a bit. For the existence part, the proof comprises two steps.

(1) We construct an outer measure from μ.

(2) This outer measure can be restricted to a measure.

Definition 2.10 (outer measure)**.** Let Ω be a domain. Then, we say that

$$\mu^* : \mathscr{P}(\Omega) \to \mathbb{R} \cup \{\infty\} \quad \text{is an outer measure} \quad \text{if}$$

(i) $\mu(\emptyset) = 0$;

(ii) if $X \subseteq Y \subseteq \Omega$, then $0 \le \mu(X) \le \mu(Y)$;

(iii) **countable subadditivity:** Given that $A = \bigcup_n A_n$, then

$$\bigcup_n \mu^*(A) \le \sum \mu^*(A).$$

The following lemma tells us how to extend our pre-measure to an outer measure.

Lemma 2.5. Suppose μ is a pre-measure defined on some algebra $\mathscr{A}$ of subsets of Ω. Then, for any $X \subseteq \Omega$, we define μ^* to be

$$\mu^*(X) = \inf \left\{ \sum_n \mu(A_n) : \{A_n\} \text{ covers } X \quad \text{and} \quad A_n \in \mathscr{A} \right\}, \quad \text{which is an outer measure.}$$

Moreover, μ^* extends μ. That is to say, for $X \in \mathscr{A}$, $\mu^*(X) = \mu(X)$.

Proof. We first prove that $\mu^*(X) \le \mu(X)$. Suppose (A_n) is a sequence in $\mathscr{A}$ such that

$$X \subseteq \bigcup_n A_n, \quad \text{then we say that } A_n \text{ is an } \mathscr{A}\text{-cover of } X.$$

Since $\emptyset \in \mathscr{A}$, we have $\mu^*(\emptyset) = 0$. Also, given any $X \subseteq Y \subseteq \Omega$, then any cover of Y using sets from $\mathscr{A}$ also covers X. By taking infimums, $\mu^*(X) \le \mu^*(Y)$.

We then let (X_n) be a sequence of subsets of Ω and

$$X = \bigcup_n X_n.$$

So, for all $\varepsilon > 0$ and each n-cover $(A_{n,m})_m$ of X_n such that

$$\sum_m \mu(A_{n,m}) \leq \mu^*(X_n) + \frac{\varepsilon}{2^n}.$$

The sets $A_{n,m}$ form a countable cover of X, i.e. similar to Cantor's proof of $\mathbb{Q}$ being countable, we shall enumerate the sets $B_1, \ldots$ in the following fashion:

$$\begin{array}{cccc} B_1 = A_{11} & B_2 = A_{12} & B_4 = A_{13} & \ldots \\ B_3 = A_{21} & B_5 = A_{22} & B_8 = A_{23} & \ldots \\ B_6 = A_{31} & B_9 = A_{32} & B_{13} = A_{33} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

Hence,

$$\mu^*(X) = \mu^*\left(\bigcup_n X_n\right) \leq \sum_{n,m} \mu\left(A_{n,m}\right) < \sum_n \left(\mu^*\left(X_n\right) + \frac{\varepsilon}{2^n}\right) = \sum_n \mu^*\left(X_n\right) + 2\varepsilon$$

Since $\varepsilon > 0$ was arbitrary, it follows that

$$\mu^*\left(\bigcup_n X_n\right) \leq \sum_n \mu^*\left(X_n\right).$$

Since for any $X \in \mathscr{A}$, the set X itself forms a cover, it follows that μ^* extends μ and so $\mu^*(X) \leq \mu(X)$.

Now, we prove that $\mu(X) \leq \mu^*(X)$. Let (A_n) be an $\mathscr{A}$-cover of X. Then, define $B_0 = A_0 \cap X$ and for $n \geq 0$, define B_{n+1} recursively as follows:

$$B_{n+1} = (A_{n+1} \cap X) \setminus \bigcup_{k=0}^{n} (A_k \cap X)$$

For example,

$$B_1 = (A_1 \cap X) \setminus (A_0 \cap X)$$
$$B_2 = (A_2 \cap X) \setminus ((A_0 \cap X) \cup (A_1 \cap X))$$

Then, $B_n \in \mathscr{A}$, the B_n are disjoint, and

$$\bigsqcup_n B_n = X.$$

and $\mu(B_n) \leq \mu(A_n)$. Since μ is a pre-measure on $\mathscr{A}$, then

$$\mu(X) = \sum_n \mu(B_n) \leq \sum_n \mu(A_n).$$

Since this holds for any $\mathscr{A}$-cover of X, then $\mu(X) \leq \mu^*(X)$. Hence, equality holds. □

Definition 2.11. We say that $S \subseteq \Omega$ satisfies Carathéodory's splitting condition with respect to the outer measure μ^* if

$$\mu^*(X) = \mu^*(X \cap S) + \mu^*(X \cap S') \quad \text{for all } X \subseteq \Omega \quad \text{and} \quad S' = \Omega \backslash S.$$

Proposition 2.2. Let $S \subseteq \mathscr{P}(\Omega)$ be the collection of subsets that satisfies Carathéodory's splitting condition. Then, S is a σ-algebra of subsets of Ω. Hence, $\sigma(\mathscr{A}) \subseteq S$ and the restriction of μ^* to the σ-algebra of $\mathscr{A}$ is a measure.

The key to the proof of the uniqueness of Carathéodory's extension theorem is the following lemma, often referred to as Dynkin's theorem:

Lemma 2.6 (Dynkin's theorem)**.** Suppose $\mathscr{A}$ is an algebra of subsets of Ω. Let $\sigma(\mathscr{A})$ be the σ-algebra generated by $\mathscr{A}$. Then $\sigma(\mathscr{A})$ is the smallest λ-system containing $\mathscr{A}$, where a collection $\mathscr{C}$ of subsets of Ω is called a λ-system if it satisfies the following conditions:

(a) $\Omega \in \mathscr{C}$

(b) **Closure under complement:** for $X \in \mathscr{C}$, we have $X' = \Omega \backslash X$ is also in $\mathscr{C}$.

(c) **Closure under disjoint union:** for a disjoint sequence $(X_n) \in \mathscr{C}$, we have $\bigcup_n X_n \in \mathscr{C}$.

Definition 2.12 (complete measure)**.** A measure μ defined on a σ-algebra $\mathscr{A}$ of subsets of Ω is complete if for all $X \subseteq U \subseteq \Omega$ with $U \in \mathscr{A}$ and $\mu(U) = 0$, we also have $X \in \mathscr{A}$ and $\mu(X) = 0$.

Lemma 2.7. Suppose μ is a complete measure defined on a σ-algebra $\mathscr{A}$ of subsets of Ω. Then,

$$\text{for all } L \subseteq X \subseteq U \subseteq \Omega \quad \text{such that} \quad L, U \in \mathscr{A} \text{ and } \mu(L) = \mu(U) < \infty,$$

we also have $X \in \mathscr{A}$ and $\mu(L) = \mu(X) = \mu(U)$.

Proof. Since $\emptyset \subseteq X \backslash L \subseteq U \backslash L$, then $\emptyset, U \backslash L \in \mathscr{A}$ and $\mu(\emptyset) = \mu(U \backslash L) = 0$. Hence, $X \backslash L \in \mathscr{A}$ and $\mu(X \backslash L) = 0$. The result follows. □

We have a nice theorem due to Maharam that has close ties with Functional Analysis. This is a deep result about the decomposability of complete measure spaces, which plays an important role in the theory of Banach Spaces (perhaps this might be covered in MA4211).

Theorem 2.4 (Maharam)**.** Every complete measure space can be decomposed into non-atomic parts, i.e. copies of products of the unit interval $[0, 1]$ on the reals, and purely atomic parts, using the counting measure on some discrete space.

2.4. *Lebesgue Measure*

As expected, we define $+$ and $-$ on $\mathbb{R}^d$ to be vector addition and subtraction respectively. There are several definitions of Lebesgue measure on $\mathbb{R}^d$. We will use the following, which is also Prof. Tran's favourite as it is close to the definition of Haar measure on a locally compact group:

Definition 2.13 (Lebesgue measure). The Lebesgue measure λ on $\mathbb{R}^d$ is the unique measure defined on a σ-algebra $\mathscr{L}(\mathbb{R}^d)$ satisfying the following properties:

(1) **open/Borel measurably:** every Borel set is measurable.

(2) **translation invariance:** for $X \in \mathscr{L}$ and $a \in \mathbb{R}^d$, we have $\lambda(a+X) = \lambda(X)$

Recall that $a+X = \{a+X : x \in X\}$. So, if $a = (a_1, \ldots, a_d)$ and $b = (b_1, \ldots, b_d)$, where $a, b \in \mathbb{R}^d$ clearly, we have $a+b = (a_1+b_1, \ldots, a_d+b_d)$

(3) **inner/outer regularity:** if $X \in \mathscr{L}$, there exist

a countable union of compact sets X_1 and a countable intersection of open sets X_2

such that $X_1 \subseteq X \subseteq X_2$ and $\lambda(X_2 \backslash X_1) = 0$. It follows that $\lambda(X_1) = \lambda(X) = \lambda(X_2)$.

(4) **normalisation on a unit cube:** $\lambda([0,1]^d) = 1$

(5) **completeness:** if $N \subseteq \mathscr{L}$ is such that $\lambda(N) = 0$, then for all $X \subseteq N$ satisfying $X \in \mathscr{L}$, we have $\lambda(X) = 0$.

If we construct a natural pre-measure on open boxes, closed boxes, etc. and use Carathéodory's extension theorem to extend it, we can obtain a measure μ satisfying all the aforementioned properties except completeness. However, we can do a completion of this measure to obtain a Lebesgue measure.

Proposition 2.3. Let $\mathscr{A} = \mathscr{A}(\mathbb{R}^d)$ be the collection of subsets of $\mathbb{R}^d$ consisting of finite unions if disjoint boxes. Then, there exists a unique pre-measure μ_0 defined on $\mathscr{A}$ such that

$$\mu_0([a_1, b_1] \times \ldots [a_d, b_d]) = \prod_{i=1}^{d} (b_i - a_i).$$

Theorem 2.5 (existence of λ). The Lebesgue measure on $\mathbb{R}^d$ exists.

Theorem 2.6. The set $\mathscr{L}$ of Lebesgue measurable subsets of $\mathbb{R}$ is a σ-algebra on $\mathbb{R}$. Also, the outer measure is a measure on $(\mathbb{R}, \mathscr{L})$.

Theorem 2.7 (Lebesgue measurable sets). Let $A \subseteq \mathbb{R}$. Then, the following are equivalent:

(a) A is Lebesgue measurable

(b) For all $\varepsilon > 0$, there exists a closed set $F \subseteq A$ such that $|A \backslash F| < \varepsilon$

(c) There exist closed sets $F_1, \ldots \subseteq A$ such that $|A \backslash \bigcup_{k=1}^{\infty} F_k| = 0$

(d) There exists a Borel set $B \subseteq A$ such that $|A \backslash B| = 0$

(e) For all $\varepsilon > 0$, there exists an open set $G \supseteq A$ such that $|G \backslash A| < \varepsilon$

(f) There exist open sets $G_1, \ldots \supseteq A$ such that $|\bigcap_{k=1}^{\infty} G_k \backslash A| = 0$

(g) There exists a Borel set $B \supseteq A$ such that $|B \backslash A| = 0$

Proof. We first prove that **(b)** implies **(c)**. Suppose **(b)** holds. Then, for every $n \in \mathbb{Z}^+$, there exists a closed set $F_n \subseteq A$ such that $|A \backslash F_n| < 1/n$. Note that

$$A \backslash \bigcup_{k=1}^{\infty} F_k \subseteq A \backslash F_n \text{ for all } n \in \mathbb{Z}^+ \quad \text{which implies} \quad \left| A \backslash \bigcup_{k=1}^{\infty} F_k \right| \le |A \backslash F_n| < \frac{1}{n}.$$

Hence, **(c)** is proven.

We then prove that **(c)** implies **(d)**. Suppose there exists a sequence of closed sets $F_1, \ldots \subseteq A$ such that $|A \backslash \bigcup_{k=1}^{\infty} F_k| = 0$. As the countable union of closed sets is Borel, then we can take the Borel set B to be the union of F_k, thus proving **(d)**.

We then prove **(d)** implies **(b)**. Then, there exists a Borel set $B \subseteq A$ such that $|A \backslash B| = 0$. Note that $A = B \cup (A \backslash B)$. Since B is a Borel set, then $B \in \mathscr{L}$ (i.e. B is Lebesgue measurable). By definition, $A \backslash B \in \mathscr{L}$ too since $A \backslash B$ has outer measure 0. Since $\mathscr{L}$ is a σ-algebra, then $\mathscr{A} \in \mathscr{L}$, proving **(b)**.

Next, we prove that **(b)** implies **(e)**. Suppose **(b)** holds. Let $A \in \mathscr{L}$ and $\varepsilon > 0$ be arbitrary. Since $\mathbb{R} \backslash A \in \mathscr{L}$ (justified as $\mathscr{L}$ is closed under complementation), there exists a closed set $F \subseteq \mathbb{R} \backslash A$ such that $|(\mathbb{R} \backslash A) \backslash F| < \varepsilon$. Note that $\mathbb{R} \backslash F$ is open (by complementation) such that $A \subseteq \mathbb{R} \backslash F$. Since

$$(\mathbb{R} \backslash F) \backslash A) = (\mathbb{R} \backslash A) \backslash F, \quad \text{then} \quad |(\mathbb{R} \backslash F) \backslash A| < \varepsilon.$$

Hence, **(e)** holds. I'm lazy to prove the remaining implications, i.e. **(e)** implies **(f)**, **(f)** implies **(g)**, and **(g)** implies **(b)** — I might re-edit the notes again if I'm happy but you can find the proofs in Axler's book. □

Example 2.26. Let $E \subseteq \mathbb{R}$. Prove that if $E \in \mathscr{L}$, then for any $\varepsilon > 0$, there exist open sets $G_1, G_2 \in \mathbb{R}$ such that $E \subseteq G_1$, $\mathbb{R} \backslash E \subseteq G_2$ and $\lambda(G_1 \cap G_2) < \varepsilon$.

Solution. We first prove the forward direction. Suppose E is Lebesgue measurable. Then, for any $\varepsilon > 0$, there exists an open set $G_1 \subseteq \mathbb{R}$ such that

$$E \subseteq G_1 \quad \text{and} \quad \lambda(G_1 \backslash E) < \varepsilon/2.$$

Since $\mathbb{R} \backslash E \in \mathscr{L}$, then there exists an open set $G_2 \subseteq \mathbb{R}$ such that

$$\mathbb{R} \backslash E \subseteq G_2 \quad \text{and} \quad \lambda(G_2 \backslash (\mathbb{R} \backslash E)) < \varepsilon/2.$$

Now, we need to prove the last result, which states that for all $\varepsilon > 0$, we have $\lambda(G_1 \cap G_2) < \varepsilon$. To see why this is true, we have

$$\begin{aligned}\lambda(G_1 \cap G_2) &= \lambda(G_1 \cap G_2 \cap E) + \lambda(G_1 \cap G_2 \cap E') \\ &\leq \lambda(G_2 \cap E) + \lambda(G_1 \cap E') \\ &= \lambda(G_2 \backslash (\mathbb{R} \backslash E)) + \lambda(G_1 \backslash E) \\ &= \varepsilon/2 + \varepsilon/2 = \varepsilon\end{aligned}$$

and we have proven all three statements. □

Example 2.27 (Axler p. 60 Question 1)**.** Prove that the set consisting of numbers in $(0,1)$ that have a decimal expansion containing one hundred consecutive 4s is a Borel subset of $\mathbb{R}$. Determine the Lebesgue measure of this set.

Solution. Let S be the mentioned set. Note that any number strictly between 0 and 1 has the following decimal expansion: $x = 0.a_1a_2a_3\ldots$, where $a_i \in \{0,1,\ldots,9\}$ for all $i \geq 1$. So, we consider all numbers x of the following form,

$$0.a_1a_2\ldots a_ja_{j+1}\ldots a_{j+99}\ldots$$

where there exists $j \geq 1$ such that $a_j = a_{j+1} = \ldots = a_{j+99} = 4$. We then define I_n to be the interval

$$I_n = (0.a_1a_2\ldots a_{n-1}444\ldots 4, 0.a_1a_2\ldots a_{n-1}444\ldots 4999\ldots)$$

which represents numbers such that starting from the n^{th} digit, have exactly 100 consecutive 4s. So,

$$S = \bigcup_{n=1}^{\infty} I_n \quad \text{where each } I_n \text{ is an open interval.}$$

Since intervals are Borel sets, and the countable union of Borel sets is also a Borel set, the first result follows.

We then determine the Lebesgue measure of this set. Note that

$$\begin{aligned}|I_1| &= 0.000999\ldots \quad \text{where there are 100 zeros after the decimal point} \\ &= 9\left(\frac{1}{10^{101}} + \frac{1}{10^{102}} + \ldots\right) = 10^{-100}\end{aligned}$$

Similarly, $|I_2| = 10^{-101}$, $|I_3| = 10^{-102}$ and so on. So, the Lebesgue measure of S, $\lambda(S)$, is the sum of the width of the intervals I_n, which evaluates to $\frac{1}{9} \cdot 10^{-99}$ (loosely speaking, the Lebesgue measure is in fact 0 but I have difficulty justifying it rigorously.). □

Example 2.28 (Axler p. 60 Question 6)**.** Suppose $A \subseteq \mathbb{R}$. Prove that A is Lebesgue measurable if and only if

$$|(-n,n) \cap A| + |(-n,n) \backslash A| = 2n$$

for every $n \in \mathbb{Z}^+$.

Solution. For the forward direction, suppose A is Lebesgue measurable. Then, $(-n,n)\cap A$ is also Lebesgue measurable and $(-n,n)\backslash A$ is also measurable. The reason that the latter holds is because the complement of a measurable set is also measurable. Since the Lebesgue measure is additive over disjoint sets, i.e. $\lambda(A\sqcup B)=\lambda(A)+\lambda(B)$, then because $(-n,n)\cap A$ and $(-n,n)\backslash A$ are disjoint, it follows that

$$\begin{aligned}|(-n,n)\cap A|+|(-n,n)\backslash A| &= \lambda((-n,n)\cap A)+\lambda((-n,n)\backslash A)\\ &= \lambda(((-n,n)\cap A))\cup((-n,n)\backslash A))\\ &= \lambda((-n,n))=2n\end{aligned}$$

For the reverse direction, we suppose that $A\subseteq\mathbb{R}$ such that $|(-n,n)\cap A|+|(-n,n)\backslash A|=2n$. We wish to prove that A is Lebesgue measurable. Since

$$\bigcup_{n=1}^{\infty}(-n,n)\quad\text{is a covering of }\mathbb{R},$$

the additivity of measure on disjoint unions implies that the condition must hold for any interval $A\subseteq\mathbb{R}$. □

Theorem 2.8 (Axler p. 60 Question 10). Let A and B be disjoint subsets of $\mathbb{R}$ such that B is Lebesgue measurable. Then,

$$|A\cup B|=|A|+|B|.$$

Note that every countable subset of $\mathbb{R}$ has outer measure 0. Recall that every outer measure μ^* satisfies countable subadditivity. Let $A=\{a_1,\ldots,a_k,\ldots\}$ be a countable subset of $\mathbb{R}$. Then, for $\varepsilon>0$, define

$$I_k=\left(a_k-\frac{\varepsilon}{2^k},a_k+\frac{\varepsilon}{2^k}\right).$$

Then, $I_1,\ldots$ is a sequence of open intervals whose union contains A, i.e.

$$\bigcup_{k=1}^{\infty}I_k\supseteq\{a_1,\ldots,a_k,\ldots\}=A.$$

So,

$$|A|\le\mu^*\left(\bigcup_{k=1}^{\infty}I_k\right)\le\sum_{k=1}^{\infty}\mu^*(I_k)=2\varepsilon\sum_{k=1}^{\infty}\frac{1}{2^k}=2\varepsilon.$$

Since ε can be made arbitrarily small, then $|A|=0$. So, a reasonable question arises is whether the converse holds, i.e. if a set A has outer measure 0, is it countable? The Cantor set provides an answer to this question. The Cantor set also gives counterexamples to other reasonable conjectures. For example, the sum of two sets with Lebesgue measure 0 can have a positive Lebesgue measure. We will discuss this in due course.

Definition 2.14. The Cantor set C is defined to be

$$[0,1]\setminus\bigcup_{n=1}^{\infty} G_n \quad \text{where } G_n = \left(\frac{1}{3},\frac{2}{3}\right) \text{ and}$$

for $n>1$, G_n is the union of the middle-third open intervals of the form

$$[0,1]\setminus\bigcup_{j=1}^{n-1} G_j.$$

Figure 1 illustrates the Cantor set can be constructed, where the top row denotes the closed unit interval $[0,1]$. The second row denotes $[0,1]\setminus(1/3,2/3)$, or rather, $[0,1]\setminus G_1$. The third row denotes $[0,1]\setminus(G_1\cup G_2)$ and so on.

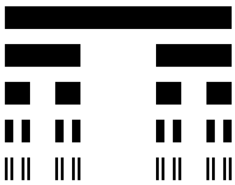


Figure 1: Construction of the Cantor set

Base 3 representations, or ternary representations, provide a useful way to think about the Cantor set. Just as $1/10 = 0.1 = 0.09999\ldots$ in the decimal representation, base 3 representations are not unique for fractions whose denominator is a power of 3. For example, $1/3 = 0.1_3 = 0.02222\ldots_3$, where the subscript 3 denotes a base 3 representation.

Notice that G_1 is the set of numbers in $[0,1]$ whose base 3 representations have 1 in the first digit after the decimal point (for those numbers that have two base 3 representations, this means both such representations must have 1 in the first digit). Also, $G_1\cup G_2$ is the set of numbers in $[0,1]$ whose base 3 representations have 1 in the first digit or the second digit after the decimal point. And so on. Hence $\bigcup_{n=1}^{\infty} G_n$ is the set of numbers in $[0,1]$ whose base 3 representations have a 1 somewhere.

Thus we have the following description of the Cantor set. In the following result, the phrase 'a base 3 representation' indicates that if a number has two base 3 representations, then it is in the Cantor set if and only if at least one of them contains no 1s. For example, both $1/3$ (which equals $0.02222\ldots_3$ and equals 0.1_3) and $2/3$ (which equals 0.2_3 and equals $0.12222\ldots_3$) are in the Cantor set.

Theorem 2.9. Let C denote the Cantor set. Then, the following hold:

(a) C is a closed subset of $\mathbb{R}$

(b) $\lambda(C) = 0$

(c) C contains no interval with more than 1 element

Proof. We first prove **(a)** by showing that it is the intersection of two closed sets. Recall that each G_n in the definition of the Cantor set can be regarded as the union of open intervals, so each G_n is an open set. Since the countable union of open sets is also open, then

$$\bigcup_{n=1}^{\infty} G_n \text{ is open.}$$

As

$$C = [0,1] \setminus \bigcup_{n=1}^{\infty} G_n,$$

we see that C is closed since its complement is open in $[0,1]$, thus proving **(a)**.

To prove **(b)**, we wish to show that its Lebesgue measure is 0. One can prove using induction that

$$\text{each } G_n \text{ is the union of } 2^{n-1} \text{ disjoint open intervals, each of length } \frac{1}{3^n} \quad \text{so} \quad |G_n| = \frac{2^{n-1}}{3^n}.$$

Since the G_n's are disjoint, then

$$|C| = |[0,1]| - \sum_{n=1}^{\infty} |G_n| = 1 - \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 0$$

which proves **(b)**. Since C has Lebesgue measure 0, then it cannot contain an interval with more than 1 element, which proves **(c)**. □

Example 2.29. Let C denote the Cantor set. We will prove that $\{x/2 + y/2 : x, y \in C\} = [0,1]$.

First, let

$$A = \{x/2 : x \in C\} \quad \text{and} \quad B = \{y/2 : y \in C\},$$

for which $\lambda(A) = \lambda(B) = 0$. Since $\lambda(C) = 0$, then $\lambda(A) = \lambda(B) = 0$ since $\lambda(A) = \lambda(B) \leq \lambda(C)$ (by a symmetric argument). We now prove that $\lambda(A+B) = 1$ as stated, which is equivalent to showing that $A + B = [0,1]$. Actually, we will first show that the sum of two Cantor sets, i.e. $C + C$, is the closed interval $[0,2]$, then re-scale to the unit interval which would suffice as an explanation.

A nice trick is to use ternary representation. Let $x, y \in C$ such that

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \quad \text{and} \quad y = \sum_{n=1}^{\infty} \frac{b_n}{3^n}.$$

Here, $a_n, b_n \in \{0,1,2\}$. Then, $x+y$ covers all of $[0,2]$. Let $z \in [0,2]$, then it has a ternary representation as follows:

$$z = \sum_{n=0}^{\infty} \frac{c_n}{3^n}, \quad \text{where } c_n = \begin{cases} 0,1 & \text{if } n = 0; \\ 0,1,2 & \text{if } n > 0. \end{cases}$$

Note that if $c_0 = 0$, then $z \in [0,1]$; otherwise, $z \in [1,2]$. It suffices to show that

$$z = \sum_{n=0}^{\infty} \frac{c_n}{3^n} = \sum_{n=1}^{\infty} \frac{a_n + b_n}{3^n} = x + y.$$

Given c_n, we can construct a_n and b_n. For example, if $c_n = 0$ and has no carry-over from $a_{n+1} + b_{n+1}$, we set $a_n = b_n = 0$; if $c_n = 0$ and has carry-over from $a_{n+1} + b_{n+1}$, we set $a_n = 2$ and $b_n = 0$. There are several other cases but we will not discuss them.

Example 2.30 (MA4262 AY24/25 Sem 1 Tutorial 3)**.** Let λ be the Lebesgue measure on $\mathbb{R}$ and $\mathscr{L}$ be the collection of Lebesgue measurable sets in $\mathbb{R}$. Are the following statements true or false? Justify your answer.

- **(a)** If $X \in \mathscr{L}$ is uncountable, then $\lambda(X) > 0$.
- **(b)** There is a closed set C which is a proper subset of $[0,1]$ such that $\lambda(C) = 1$.
- **(c)** If X are Y are Lebesgue measurable, then $\lambda(X \cup Y) + \lambda(X \cap Y) = \lambda(X) + \lambda(Y)$.
- **(d)** There is a closed subset $C \subseteq ([0,1] \backslash \mathbb{Q})$ with $\lambda(C) > 0$.

Solution.

- **(a)** False. Consider the Cantor set C which is uncountable but $\lambda(C) = 0$
- **(b)** True. Consider $[0,1]$ and remove all the rational points (the rationals form a countable set. Label the resulting set C, which is closed since it contains all its limit points. Since the measure of irrationals in $[0,1]$ is 1, the result follows by construction.
- **(c)** True since $\lambda(X \cup Y) = \lambda(X) + \lambda(Y) - \lambda(X \cap Y)$.
- **(d)** True, see **(b)** for something similar. □

Definition 2.15. A subset of $\mathbb{R}$ is F_σ if it is a countable union of closed subsets of $\mathbb{R}$ (closed can be here replaced by compact). A subset of $\mathbb{R}$ is G_δ if it is a countable intersection of open subsets of $\mathbb{R}$.

Theorem 2.10 (MA4262 AY24/25 Sem 1 Tutorial 3)**.** A closed set is G_δ and an open set is F_σ.

Proof. We first prove that a closed set is G_δ. Let $[a,b]$ be an arbitrary closed subset of $\mathbb{R}$. We can write

$$[a,b] = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b + \frac{1}{n}\right) \quad \text{which is a countable intersection of open subsets of } \mathbb{R}.$$

We then prove that an open set is F_σ. Let (a,b) be an arbitrary open subset of $\mathbb{R}$. We can write

$$(a,b) = \bigcup_{n=1}^{\infty} \left[a - \frac{1}{n}, b + \frac{1}{n}\right] \quad \text{which is a countable union of closed subsets of } \mathbb{R}.$$

The result follows. □

Theorem 2.11 (MA4262 AY24/25 Sem 1 Tutorial 3)**.** $\mathbb{Q}$ is F_σ but not G_δ.

Proof. Let $q \in \mathbb{Q}$. Then, $\{q\}$ since singletons are closed sets. So,

$$\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\} \quad \text{which is a countable union of closed subsets of } \mathbb{R}.$$

It follows that $\mathbb{Q}$ is F_σ. We then prove that $\mathbb{Q}$ is not G_δ. This follows from the fact that $\mathbb{Q}$ is the countable union of nowhere-dense sets (we say that such sets are meagre from a topological point-of-view). □

Theorem 2.12 (MA4262 AY24/25 Sem 1 Tutorial 3)**.** There is a Borel set which is neither F_σ nor G_δ.

Proof. Consider the complement of the Cantor set C in $[0,1]$. □

Example 2.31 (MA4262 AY24/25 Sem 1 Tutorial 3)**.** Let λ be the Lebesgue measure on $\mathbb{R}^2$ and $\mathscr{L}$ be the collection of Lebesgue measurable sets in $\mathbb{R}^2$. Let $f:\mathbb{R}^2\to\mathbb{R}^2,(x,y)\mapsto(y,x)$ be reflection by the line $x=y$. Without using the fact that continuous functions are measurable, show that

(a) X is F_σ if and only if $f(X)$ is F_σ;

(b) X is Lebesgue measurable if and only if $f(X)$ is Lebesgue measurable.

Solution.

(a) Suppose X is F_σ. Then,

$$X=\bigcup_{n=1}^{\infty}C_n \quad \text{where } C_n \text{ is closed in } \mathbb{R}^2 \text{ for all } n\in\mathbb{N}.$$

Hence,

$$f(X)=f\left(\bigcup_{n=1}^{\infty}C_n\right)=\bigcup_{n=1}^{\infty}f(C_n).$$

We need to prove that each $f(C_n)$ is closed in $\mathbb{R}^2$. Note that f is bijective and continuous (i.e. a homeomorphism) so the image of any closed set under f is also closed. Hence, $f(C_n)$ is closed in $\mathbb{R}^2$. The proof of the reverse direction is similar as one would need to use the fact that $f:\mathbb{R}^2\to\mathbb{R}^2$ is a homeomorphism.

(b) Suppose X is Lebesgue measurable. Then, for all $\varepsilon>0$, there exists an open set $G\supseteq X$ such that $|G\backslash X|<\varepsilon$. Since f is a homeomorphism, then $f(G)$ is open and $f(X)\subseteq f(G)$. Also, since f is measure-preserving, then $\lambda(f(G\backslash X))=\lambda(G\backslash X)<\varepsilon$, which shows that $f(X)$ is also Lebesgue measurable. Similar to **(a)**, the reverse direction of the proof uses the fact that f is invertible. □

Example 2.32 (MA4262 AY24/25 Sem 1 Tutorial 3)**.** Let $X\subseteq\mathbb{R}$ such that $\lambda(X)>0$. Prove that for each $\alpha\in(0,1)$, there exists an open interval $I=(a,b)$ so that $\lambda(X\cap I)\geq\alpha\lambda(I)$. Loosely speaking, X contains almost a whole interval.

(Hint: Choose an open set U that contains X, and such that $\lambda(X)\geq\alpha\lambda(U)$. Write U as the countable union of disjoint open intervals, and show that one of these intervals must satisfy the desired property.)

Solution. Note that $\lambda(I)=b-a$. Let $U\supseteq X$ be an open set such that $\lambda(X)\geq\alpha\lambda(U)$. We then write

$$U=\bigsqcup_{n=1}^{\infty}(a_n,b_n) \quad \text{so} \quad \lambda(U)=\sum_{n=1}^{\infty}(b_n-a_n).$$

So,

$$\alpha\lambda(U) \le \lambda(X) = \lambda(X \cap U) = \lambda\left(X \cap \bigsqcup_{n=1}^{\infty}(a_n, b_n)\right) = \sum_{n=1}^{\infty}\lambda(X \cap (a_n, b_n)) = \sum_{n=1}^{\infty}\lambda(X \cap I_n).$$

Here, $I_n = (a_n, b_n)$. Suppose on the contrary that for all intervals $I_n = (a_n, b_n)$, we have $\lambda(X \cap I_n) < \alpha\lambda(I_n)$. Then, summing over all n, we have

$$\sum_{n=1}^{\infty}\lambda(X \cap I_n) < \alpha\sum_{n=1}^{\infty}\lambda(I_n) = \alpha\lambda(U),$$

which is a contradiction. □

Definition 2.16 (dyadic cubes). The dyadic cubes are a collection of cubes in $\mathbb{R}^n$ of different sizes such that the set of cubes of each scale partition $\mathbb{R}^n$ and each cube in one scale may be written as a union of cubes of a smaller scale.

Proposition 2.4 (construction of dyadic cubes). In Euclidean space, we can construct dyadic cubes as follows. For each $k \in \mathbb{Z}$, let Δ_k be the set of dyadic cubes in $\mathbb{R}^n$ of side length 2^{-k} and corners in the set

$$2^{-k}\mathbb{Z}^n = \left\{2^{-k}(v_1, \ldots, v_n) : v_j \in \mathbb{Z}\right\}.$$

Let $\Delta = \bigcup_{k=1}^{\infty}\Delta_k$.

Theorem 2.13. Here are some important features of dyadic cubes.

(a) For each $k \in \mathbb{Z}$, Δ_k partitions $\mathbb{R}^n$.

(b) All cubes in Δ_k have the same side length, namely 2^{-k}.

(c) If the interiors of two cubes $Q, R \in \Delta$ have non-empty intersection, then either $Q \subseteq R$ or $R \subseteq Q$.

(d) Each $Q \in \Delta_k$ may be written as a union of 2^n cubes in Δ_{k+1} with disjoint interiors.

2.5. *Product Measure*

In Probability Theory, one often needs to consider the space obtained from taking an infinite product of a certain space (e.g. $\{H, T\}^{\mathbb{N}}$). One would like to do analysis on such space, which calls for a rigorous definition of measure.

Definition 2.17 (product σ-algebra and product measure). Let

$$(\Omega_1, \mathscr{A}_1, \mu_1) \quad \text{and} \quad (\Omega_2, \mathscr{A}_2, \mu_2) \quad \text{be measure spaces.}$$

The product σ-algebra $\mathscr{A} = \mathscr{A}_1 \otimes \mathscr{A}_2$ is the smallest σ-algebra containing $A_1 \times A_2$, where $A_1 \in \mathscr{A}_1, A_2 \in \mathscr{A}_2$. Also, the product measure $\mu = \mu_1 \otimes \mu_2$ is the unique measure defined on $\mathscr{A}_1 \otimes \mathscr{A}_2$

such that

$$\mu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2) \quad \text{for all } A_1 \in \mathscr{A}_1, A_2 \in \mathscr{A}_2$$

Lemma 2.8. Let $\mathscr{A}$ be the collection of unions of finitely-many disjoint rectangles, i.e.

$$\text{sets of the form } A_1 \times A_2, \quad \text{where } A_1 \in \mathscr{A}_1, A_2 \in \mathscr{A}_2.$$

Then, $\mathscr{A}$ is an algebra and $\mathscr{A}_1 \otimes \mathscr{A}_2$ is the σ-algebra generated by $\mathscr{A}$.

Lemma 2.9. There exists a unique function $\mu_0 : \mathscr{A} \to \mathbb{R}^+$ such that

$$\mu_0(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2) \quad \text{and} \quad \mu_0(A \sqcup A') = \mu(A) + \mu(A').$$

Lemma 2.10 (replacement of compactness)**.** If $A^{(1)} \supseteq A^{(2)} \supseteq \ldots \in \mathscr{A}$ such that

$$\bigcap_n A^{(n)} = \emptyset \quad \text{then we have} \quad \lim_{n \to \infty} \mu_0\left(A^{(n)}\right) = 0.$$

Theorem 2.14. When Ω_1, Ω_2 are σ-finite, then the product measure $\mu = \mu_1 \otimes \mu_2$ exists.

Definition 2.18 (cross-section)**.** Let X and Y be sets and suppose $E \subseteq X \times Y$. Then, for any $a \in X$ and $b \in Y$, we define the cross-sections $[E]_a$ and $[E]^b$ as follows (relate to fiber):

$$[E]_a = \{y \in Y : (a,y) \in E\} \quad \text{and} \quad [E]^b = \{x \in X : (x,a) \in E\}$$

Example 2.33. Suppose X and Y are sets such that $X \times Y$ represents an ellipse in $\mathbb{R}^2$ (this is merely for illustration purposes). Figure 2 depicts some of the cross-sections of $X \times Y$.

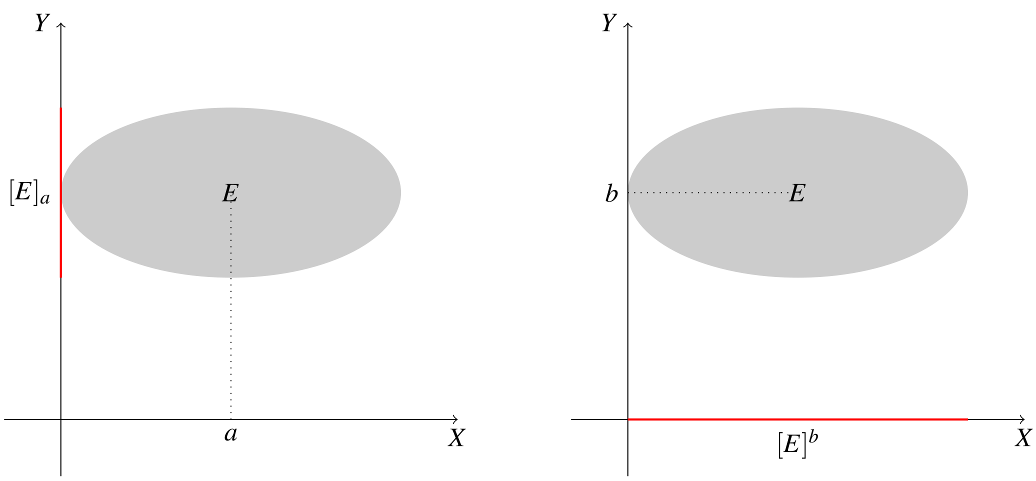


Figure 2: Some cross-sections of $X \times Y$

Example 2.34. Suppose X and Y are sets and $A \subseteq X$ and $B \subseteq Y$. If $a \in X$ and $b \in Y$, then

$$[A\times B]_a = \begin{cases} B & \text{if } a \in A, \\ \emptyset & \text{if } a \notin A \end{cases} \quad \text{and} \quad [A\times B]^b = \begin{cases} A & \text{if } b \in B, \\ \emptyset & \text{if } b \notin B, \end{cases}$$

We will only verify for the first piece-wise equation, i.e. involving $[A\times B]_a$. By definition, $[A\times B]_a = \{y \in Y : (a,y) \in A\times B\}$. Note that if $a \in A$, then $a \in X$, so we must have $y \in B$. On the other hand, if $a \notin A$, then $(a,y) \notin A\times B$.

Definition 2.19 (infinite product measure)**.** Let $(\Omega_n, \mathscr{A}_n, \mu_n)$ be a sequence of measure spaces. Then, define

$$\bigotimes_{n\in\mathbb{N}} \mathscr{A}_n = \sigma\left(\{\mathscr{A}_1 \times \ldots \times \mathscr{A}_n \times \Omega_{n+1} \times \ldots : A_i \in \mathscr{A}_i \text{ for some } i \in \mathbb{N}\}\right).$$

Note that this σ-algebra can also be generated by $\Omega_1 \times \ldots \times A_n \times \Omega_{n+1} \times \ldots$, etc.

Theorem 2.15. The infinite product measure exists and is unique.

Proof. Use Carathéodory's extension theorem. □

Definition 2.20 (infinite probability measure)**.** Extending to Probability theory, a measure space $(\Omega, \mathscr{F}, P)$ is a probability space if $P(\Omega) = 1$. When $(\Omega_n, \mathscr{F}_n, P_n)$ is a sequence of probability spaces, then we define

$$\bigotimes_{n\in\mathbb{N}} P_n\left(A_1 \times \ldots \times A_n \times \Omega_{n+1} \times \ldots\right) = \prod_{i=1}^{n} \mu_i(A_i).$$

3. Integration

Wish to exchange limit and integral. The main topics discussed are the monotone convergence theorem (MCT), Fatou's lemma, and the dominated convergence theorem (DCT).

Definition 3.1 (measurable function)**.** Let $(\Omega, \mathscr{A}, \mu)$ be a measure space. A function $f : \Omega \to \mathbb{R}$ is said to be measurable if for all $\alpha \in \mathbb{R}$,

$$L_f^{\geq}(\alpha) = \{x \in X : f(x) \geq \alpha\} \quad \text{is measurable.}$$

Also, a function $f : \Omega \to \mathbb{R} \cup \{\pm\infty\}$ is measurable if

$$f^{-1}(-\infty) \text{ is measurable} \quad \text{and} \quad L_f^{\geq}(\alpha) \text{ is measurable for } \alpha \in [-\infty, \infty].$$

Lemma 3.1 (integrability of characteristic function)**.** Let $(\Omega, \mathscr{A}, \mu)$ be a measure space, $A \subseteq \Omega$ and

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A; \\ 0 & \text{if } x \notin A \end{cases} \quad \text{denote its characteristic function.}$$

Then, $\mathbf{1}_A$ is measurable if and only if A is measurable.

Proof. Note that

$$L_\alpha^{\geq}(\mathbf{1}_A) = \{x \in X : f(\alpha) \geq \mathbf{1}_A\} = \begin{cases} A & \text{if } \alpha \leq 1; \\ \emptyset & \text{if } \alpha > 1. \end{cases}$$

and the result follows. □

Lemma 3.2. Let $(\Omega, \mathscr{A}, \mu)$ be a measure space. Then, the following are equivalent for any $f : \Omega \to \mathbb{R}$:

(i) f is measurable

(ii) for all $\alpha \in \mathbb{R}$, we have $f^{-1}([\alpha, \infty)) \in \mathscr{A}$ is measurable

(iii) for all $\alpha \in \mathbb{R}$, we have $f^{-1}((\alpha, \infty)) \in \mathscr{A}$ is measurable

(iv) for all $\alpha \in \mathbb{R}$, we have $f^{-1}(-\infty, \alpha] \in \mathscr{A}$ is measurable

(v) for all $\alpha \in \mathbb{R}$, we have $f^{-1}(-\infty, \alpha) \in \mathscr{A}$ is measurable

(vi) For all Borel subsets $X \subseteq \mathbb{R}$, $f^{-1}(X)$ is measurable

Proof. We will only prove **(i)** implies **(ii)**. To see why, let f be measurable. Then,

$$L_f^{\geq}(\alpha) = \{x \in X : f(x) \geq \alpha\} = f^{-1}([\alpha, \infty)) \quad \text{so } \textbf{(ii)} \text{ follows.}$$

Might write the proofs of the other implications if I feel like it. □

Lemma 3.3. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a continuous function. Then, it is measurable.

Proof. It suffices to prove that if f is continuous, then $f^{-1}((\alpha,\infty))$ is measurable. This is true because $f^{-1}((\alpha,\infty))$ is open and hence Borel. □

Lemma 3.4. Let $(\Omega,\mathscr{A},\mu)$ be a measure space. Suppose f and g are real-valued measurable functions. Then, we have the following:

(a) $f+g$ is measurable
(b) cf is measurable for $c \in \mathbb{R}$
(c) fg is measurable
(d) $|f|$ is measurable
(e) $\max\{f,g\}$ is measurable

Proof. We first prove **(a)**. Here's a rough sketch. Consider

$$\{x : f(x) > t\} \cap \{x : g(x) > \alpha - t\}.$$

This is the intersection of measurable sets, which is also measurable. If we know that $f(x) > t$ and $g(x) > \alpha - t$, then $f(x) + g(x) > \alpha$. Putting everything together, we have

$$\{x : f(x) + g(x) > \alpha\} = \bigcup_{q \in \mathbb{Q}} \{x : f(x) > q\} \cap \{x : g(x) > \alpha - q\}$$

Since this is the countable union of measurable sets, then **(a)** follows. Showing that the above is an equality is left as an exercise in class. I'll prove that in this set of notes. We have RHS $\subseteq$ LHS being trivial; LHS $\subseteq$ RHS follows from the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ (need to avoid union over all $r \in \mathbb{R}$ since this would yield an uncountable union).

To prove **(d)**, we need to use the definition of the absolute value function; to prove **(c)**, we need to invoke the identity

$$fg = \frac{(f+g)^2 - f^2 - g^2}{2}$$

and **(e)** follows by considering the following intersection:

$$\{x \in X : f(x) \le \alpha\} \cap \{x \in X : g(x) \le \alpha\}$$

since $\max\{f(x), g(x)\} \le \alpha$ if and only if $f(x) \le \alpha$ and $g(x) \le \alpha$. □

Definition 3.2 (pointwise limit). Let Ω be a measure space and $f_n : \Omega \to \mathbb{R}$ be a sequence of functions. We say that $f : \Omega \to \mathbb{R}$ is the pointwise limit of f_n if for all $x \in \Omega$, we have $f_n(x) \to f(x)$.

Definition 3.3. We say

$$f = \sup f_n \text{ if } f(x) = \sup f_n(x) \quad \text{and} \quad f = \inf f_n \text{ if } f(x) = \inf f_n(x).$$

Also,

$$f = \limsup f_n \text{ if } f(x) = \lim_{m\to\infty} \sup_{n\geq m} f_n(x) \quad \text{and} \quad f = \liminf f_n \text{ if } f(x) = \lim_{m\to\infty} \inf_{n\geq m} f_n(x)$$

Lemma 3.5. Suppose f is a pointwise limit of f_n. Then,

$$f = \limsup f_n = \liminf f_n.$$

Conversely, if $\limsup f_n = \liminf f_n$ is a function f, then f is the pointwise limit of f_n.

Lemma 3.6. Suppose $(\Omega, \mathscr{A}, \mu)$ is a measure space and f_n is a sequence of measurable functions on Ω. Then, the following hold for $f : \Omega \to \mathbb{R}$:

- **(i)** If $f = \sup f_n$, then f is measurable
- **(ii)** If $f = \inf f_n$, then f is measurable
- **(iii)** If $f = \limsup f_n$, then f is measurable
- **(iv)** If $f = \liminf f_n$, then f is measurable
- **(v)** If f is the pointwise limit of f_n, then f is measurable

Proof. Note that **(ii)**, **(iii)**, and **(iv)** follow from **(i)**. To prove **(i)**, we note that f_n is a sequence of measurable functions so

$$\bigcap_{n\in\mathbb{N}} \{x \in \Omega : f_n(x) \leq \alpha\} \quad \text{is measurable.}$$

As such, $\{x \in \Omega : f(x) \leq \alpha\}$ is measurable, which concludes that $f^{-1}(-\infty, \alpha]$ is measurable. In fact, **(v)** follows from **(i)** as $f = \lim f_n$ so $f = \limsup f_n$. □

Definition 3.4 (almost everywhere)**.** We say that a property holds almost everywhere on Ω if there exists $N \in \mathscr{A}$ with $\mu(N) = 0$ such that for all $x \in \Omega \backslash N$, $P(x)$. We abbreviate this as $P(x)$ a.e., or P holds a.e. for all $x \in \Omega$.

Example 3.1. f_n converges pointwise to f almost everywhere. This is equivalent to saying that f_n converges almost everywhere to f.

Example 3.2. $f = g$ a.e. is equivalent to saying that $f = g$ except on some null set $N \in \mathscr{A}$ such that $\mu(N) = 0$.

Example 3.3. Let $M^+(\Omega, \mathscr{A})$ denote the collection of non-negative measurable functions. If

$$f \in M^+(\Omega, \mathscr{A}, \mu) \text{ and } \int f = 0, \quad \text{is it true that } f = 0 \text{ a.e.?}$$

Solution. Yes. Suppose on the contrary that the statement is false. Then, for all $N \in \mathscr{A}$, we have $\mu(N) > 0$, which implies

$$\int f\, d\mu \geq \int_N f\, d\mu > 0.$$

This is a contradiction. □

Corollary 3.1. Suppose f_n is a sequence of functions and f_n converges to f almost everywhere. Then, there exists a measurable function

$$\widetilde{f} = f \quad \text{almost everywhere}$$

such that f_n converges to $\widetilde{f}$. Furthermore, if μ is a complete measure, then f is measurable.

Proof. There exists a null set $N \in \mathscr{A}$ where $\mu(N) = 0$ such that $x \in \Omega \backslash N$ and $f_n \to f$. Set

$$g_n = \begin{cases} f_n & \text{if } x \in \Omega \backslash N; \\ 0 & \text{if } x \in N. \end{cases} \quad \text{which implies} \quad g = \begin{cases} f(x) & \text{if } x \notin N; \\ 0 & \text{if } x \in N. \end{cases}$$

Since g_n is measurable, then g is measurable. Note that $g = f$ almost everywhere, $g_n \to g$ pointwise, $f_n = g_n$ almost everywhere, and $f_n \to g$ almost everywhere. Hence, we have proven the first part of the corollary by choosing $\widetilde{f} = g$. The second part is trivial □

Definition 3.5 (simple function)**.** A function $f : \Omega \to \mathbb{R}$ is simple if there exist finitely many disjoint sets $E_1, \ldots, E_n$ such that $\Omega = \bigsqcup E_i$ and on each E_i, f is a constant.

Lemma 3.7. Suppose f is a simple function in standard form as defined above. Then,

$$f \text{ is measurable} \quad \text{if and only if} \quad E_i \text{ is measurable for all } 1 \leq i \leq n.$$

Proposition 3.1. Suppose f is measurable. Then there exists a sequence of measurable simple functions ϕ_n which converges pointwise to f. Moreover, if $f \geq 0$, then $\phi_n \leq \phi_{n+1}$.

Proof. We first prove the case where f is restricted to be non-negative. Then, let n be arbitrary. Set

$$\phi_n(x) = \begin{cases} \frac{k}{2^n} & \text{if } f(x) \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right), \text{ where } k \in \mathbb{Z}_{\geq 0} \text{ and } \frac{k}{2^n} \leq 3n; \\ 0 & \text{otherwise .} \end{cases}$$

Note that ϕ_n is measurable as

$$\phi_n^{-1}(\alpha, \infty] \quad \text{is either} \quad f^{-1}(\lfloor \alpha \rfloor, \infty) \text{ or } f^{-1}(\lfloor \alpha \rfloor - 1, \infty).$$

By construction, $\phi_n \leq \phi_{n+1} \leq f$ so $|f \cdot \phi_n| < 1/2^k$ for all $x \leq n$. Hence, $\phi_n \to f$ pointwise.

For general f, let

$$f^+ = \max\{f, 0\} \quad \text{and} \quad f^- = \min\{f, 0\}.$$

Then, f^+ and $f^-$are measurable. We obtain sequences of functions ϕ_n^+ and ϕ_n^- which converge to f^+ and f^- pointwise, where $\phi_n^+, \phi_n^- \geq 0$. Finally, $\phi_n = \phi_n^+ - \phi_n^-$, which converges pointwise to f. □

Example 3.4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Show that $f'(x)$ is a Lebesgue measurable function. *Hint:* Write $f'(x)$ as a limit of a sequence of functions.

Solution. Since f is differentiable, then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{n\to\infty} n\left[f\left(x + \frac{1}{n}\right) - f(x)\right].$$

Define

$$f_n(x) = n\left[f\left(x + \frac{1}{n}\right) - f(x)\right].$$

Each function $f_n(x)$ is measurable because it is constructed from f, which is differentiable (hence continuous, and thus measurable). As the pointwise limit of a sequence of measurable functions is also measurable, the result follows. □

Definition 3.6 (uniform convergence)**.** Recall that f_n is said to converge uniformly to f if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|f_n - f| < \varepsilon$.

Recall from MA3210 that a sequence of functions that converges pointwise need not converge uniformly. However, Egorov's theorem (which we will state in just a bit) mentions that

a pointwise sequence of functions on a measure space with finite total measure converges uniformly.

This means that the sequence converges uniformly except on a set that can have arbitrarily small measure.

Theorem 3.1 (Egorov)**.** Suppose f_n is a sequence of measurable functions that converge pointwise to f. Then, for every $\varepsilon > 0$, there exists $E \in \mathscr{A}$ such that $\mu(E) < \varepsilon$. On $\Omega \backslash E$, $f_n \to f$ uniformly (here, $\mu(\Omega) < \infty$).

Remark 3.1. In Egorov's theorem, E is compact.

Proof. For all $k, n > 0$, define

$$E_{n,k} = \left\{x \in \Omega : |f_m - f| > \frac{1}{k}\right\} \quad \text{for some } m \geq n.$$

We claim that $\bigcap_n E_{n,k} = \emptyset$. To see why, we have $f_n \to f$ pointwise. So, if we fix x, then there exists some n such that for all $m \geq n$, we have $|f_m - f| < 1/k$. Also, note that $E_{n+1}, k \subseteq E_{n,k}$. As $\mu(\Omega) < \infty$

we have

$$0 = \mu\left(\bigcap_n E_{n,k}\right) = \inf \mu\left(E_{n,k}\right).$$

So, there exists n_k such that

$$\mu\left(E_{n_k}, k\right) < \frac{\varepsilon}{2^k}.$$

Finally, set

$$E = \bigcup_k E_{n_k,k} \quad \text{which implies} \quad \mu(E) < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{8} + \ldots \leq \varepsilon.$$

□

Theorem 3.2 (Lusin). Suppose f is measurable on $\left([0,1]^d, \mathscr{L}_{[0,1]^d}, \lambda\right)$. Then, there exists

a compact set $K \subseteq \Omega$ such that $\lambda(\Omega \backslash K) < \varepsilon$ and f is continuous on K w.r.t. subspace topology.

Axler mentions that Lusin's theorem is surprising (p. 66). It also does not invoke the terminology 'subspace topology'. It views Lusin's theorem as follows: an arbitrary Borel measurable function is almost continuous, in the sense that its restriction to a large closed set is continuous. Here, the phrase *large closed set* means that we can take the complement of the closed set to have arbitrarily small measure.

Proof. First, we consider the case where f is a characteristic function of a measurable set. Then, $f = \mathbf{1}_A$, where A is measurable. By inner regularity, we have

$$\begin{aligned} K_1 \subseteq A \text{ is compact} &\quad \text{such that} \quad \mu(A \backslash K_1) < \varepsilon/2 \quad \text{and} \\ K_2 \subseteq \Omega \backslash A \text{ is compact} &\quad \text{such that} \quad \mu((\Omega \backslash A) \backslash K_2) < \varepsilon/2 \end{aligned}$$

Now, set $K = K_1 \cup K_2$. We claim that f is continuous on K. Note that f is continuous on K_1 and f is continuous on K_2. Since K_1 and K_2 are disjoint, then $d(K_1, K_2) > 0$ and K_1 and K_2 are disconnected components of K.

For simple functions, we do something similar. We now consider the general case. By the earlier result, we obtain a sequence of simple functions ϕ_n which converges pointwise to f. We get a compact subset $K \subseteq \Omega$ such that $\mu(\Omega \backslash K) < \varepsilon/2$ and $\phi_n \to f$ uniformly.

For all ϕ_n, get K_n such that $\mu(\Omega \backslash K_n) < \varepsilon_n$ such that $\phi_n|_{K_n}$ is continuous. Assume that $K = K_1 = K_2 = \ldots$, then on K, we see that $\phi_n|_K$ is a sequence of continuous functions converging uniformly to $f|_K$. Since uniform convergence preserves continuity, then $f|_K$ is continuous. □

Theorem 3.3 (Tietze extension theorem). Suppose we have a compact subset $K \subseteq [-m,m]^d$ and g is continuous on K, then g can be extended to $\widetilde{g}$ which is continuous on $[-m,m]^d$.

Corollary 3.2. Suppose f is measurable on $(\mathbb{R}^d, \mathscr{L}, \lambda)$. Then, f is the pointwise limit of continuous functions.

Proof. Note that

$$f_m = f\mathbf{1}_{[-m,m]^d} \to f \text{ pointwise.}$$

If we can construct a sequence $(f_{m,n})$ of continuous functions such that $f_{m,n} \to f_m$ pointwise, then $f_{n,n} \to f$ pointwise. Choose a sequence of subsets $K_{m,n} \subseteq [-m,m]^d$ such that

$$\mu([-m,m]^d \backslash K_{m,n}) < \frac{1}{n}$$

and f is continuous on $K_{m,n}$ with respect to the subspace topology. Set $f_{m,n}$ to be a continuous extension of $f|_{K_{m,n}}$. Then, one checks that $f_{m,n} \to f$ almost everywhere. □

3.1. *Definition of Integration*

Definition 3.7. Suppose $f = \mathbf{1}_A$, where A is measurable. Its integration on Ω with respect to μ,

$$\text{denoted by } \int_\Omega f \, d\mu, \quad \text{is defined to be} \quad \mu(A).$$

We say that f is integrable if $\mu(A) < \infty$.

Example 3.5. The characteristic function of Cantor's middle third set is Riemann integrable.

Definition 3.8 (Lebesgue integral). Suppose $f \geq 0$ is a simple function in standard form, i.e. there exist $(E_i)_{i=1}^n$ which are disjoint measurable subsets of Ω such that $f(E_i) = c_i$ and $E_i = f^{-1}(c_i)$. Then, its integration,

$$\int f \, d\mu, \quad \text{is defined to be} \quad \int f \, d\mu = \sum_{i=1}^n c_i \mu(E_i).$$

Remark 3.2. Easier to partition the range, i.e. Lebesgue integral is better than Riemann integral.

As quoted by Lebesgue, "*I have to pay a certain sum, which I have collected in my pocket. I take the bills and coins out of my pocket and give them to the creditor in the order I find them until I have reached the total sum. This is the Riemann integral. But I can proceed differently. After I have taken all the money out of my pocket I order the bills and coins according to identical values and then I pay* *the several heaps one after the other to the creditor. This is my integral.*" So, Lebesgue's theory can be thought of as partitioning the range, whereas Riemann's theory depends on partitioning the domain of the function.

Example 3.6 (Axler p. 84 Question 2). Suppose X is a set, $\mathscr{S}$ is a σ-algebra on X and $c \in X$. Define the Dirac measure δ_c on $(X, \mathscr{S})$ by

$$\delta_c(E) = \begin{cases} 1 & \text{if } c \in E; \\ 0 & \text{if } c \notin E. \end{cases}$$

Prove that if $f : X \to [0, \infty]$ is $\mathscr{S}$-measurable, then

$$\int f \, d\delta_c = f(c).$$

Solution. We use the definition of the Lebesgue integral to obtain

$$\int f \, d\delta_c = \sum_{x \in X} f(x)\delta_c(\{x\}) = f(c)\delta_c(\{c\}) = f(c).$$

□

Definition 3.9 (integrability). Suppose f is measurable on $(\Omega, \mathscr{A}, \mu)$ and $f \geq 0$. Its integral

$$\int f \, d\mu \quad \text{is defined to be} \quad \sup\left\{\int \phi \, d\mu : \phi \text{ simple function and } 0 \leq \phi \leq f\right\}.$$

The function is integrable if the supremum is finite.

Definition 3.10. Suppose f is measurable on $(\Omega, \mathscr{A}, \mu)$ and $f^+ = \max\{f, 0\}$ and $f^- = \min\{f, 0\}$. Define

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu.$$

If both

$$\int f^+ \, d\mu \text{ and } \int f^- \, d\mu \quad \text{exist,} \quad \text{then} \quad f \text{ is integrable;}$$

otherwise, f is not integrable.

Example 3.7. Prove or disprove: All nonnegative Lebesgue measurable functions from $\mathbb{R} \to \mathbb{R}$ are Lebesgue integrable (i.e. $\int f d\lambda < +\infty$).

Solution. The statement is false. Consider $f(x) = 1$ for all $x \in \mathbb{R}$, for which $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable but it is not Lebesgue integrable since the integral is infinite. □

3.2. *Monotone Convergence Theorem*

Theorem 3.4 (monotone convergence theorem)**.** Suppose $(\Omega, \mathscr{A}, \mu)$ is a measure space. Say ϕ_n is a sequence of measurable simple functions such that $\phi_n \leq \phi_{n+1}$, (i.e. for all x, $\phi_n(x) \leq \phi_{n+1}(x)$) which converges pointwise to a measurable function f. Then,

$$\lim_{n\to\infty} \int \phi_n \, d\mu = \int f \, d\mu.$$

Example 3.8. Define $M^+(\omega, \mathscr{A})$ to be the collection of non-negative measurable functions. Let (f_n) be a sequence of in $M^+(\Omega, \mathscr{A}, \mu)$. Prove that

$$\int \left(\sum_{n=0}^{\infty} f_n \right) d\mu = \sum_{n=0}^{\infty} \left(\int f_n d\mu \right).$$

Solution. Note that each f_n is a non-negative measurable function. To see why the equation is true, we have

$$\begin{aligned} \int \left(\sum_{n=0}^{\infty} f_n \right) d\mu &= \int \left(\lim_{N\to\infty} \sum_{n=0}^{N} f_n \right) d\mu \\ &= \lim_{N\to\infty} \int \sum_{n=0}^{N} f_n \, d\mu \quad \text{by monotone convergence theorem} \\ &= \lim_{N\to\infty} \left(\int f_0 \, d\mu + \ldots + \int f_N \, d\mu \right) \quad \text{by linearity of integration} \\ &= \lim_{N\to\infty} \sum_{n=0}^{N} \int f_n \, d\mu \end{aligned}$$

and the result follows. □

Example 3.9 (Axler p. 85 Question 10)**.** Suppose $(X, \mathscr{S}, \mu)$ is a measure space and $f_1, f_2, \ldots$ is a sequence of nonnegative $\mathscr{S}$-measurable functions. Define $f : X \to [0, \infty]$ by

$$f(x) = \sum_{k=1}^{\infty} f_k(x).$$

Prove that

$$\int f \, d\mu = \sum_{k=1}^{\infty} \int f_k \, d\mu.$$

Solution. We have

$$\int \sum_{k=1}^{n} f_k \, d\mu = \sum_{k=1}^{n} \int f_k \, d\mu.$$

Define

$$g_n = \sum_{k=1}^{n} f_k.$$

Since $f_1, f_2, \ldots$ is a sequence of non-negative $\mathscr{S}$-measurable functions, then $g_{n+1} \geq g_n$. Also,

$$g_n \to \sum_{k=1}^{\infty} f_k \quad \text{pointwise.}$$

By the monotone convergence theorem,

$$\lim_{n\to\infty} \int g_n \, d\mu = \int \sum_{k=1}^{\infty} f_k \, d\mu$$
$$\lim_{n\to\infty} \sum_{k=1}^{n} \int f_k \, d\mu = \int f \, d\mu$$

and the result follows. □

Lemma 3.8. The following hold:

(i) Sum of simple functions is simple;

(ii) Product of simple functions is simple;

(iii) If f is simple, then for any $c \in \mathbb{R}$, we have cf is simple

Proof. We only prove **(i)**. Let f and g be simple functions. Then, there exist $(E_i)_{i=1}^n, (E'_j)_{j=1}^n$ such that f is constant on each E_i and g is constant on each E'_j. On $E_i \cap E'_j$, f is a simple function; the same can be said for g, and the result follows. □

Example 3.10. Let $L(\mathbb{R}, \mathscr{L}, \lambda)$ be the set of Lebesgue integrable functions over the measure space $(\mathbb{R}, \mathscr{L}, \lambda)$. If $f \in L(\mathbb{R}, \mathscr{L}, \lambda)$, then

$$\int_a^b f(x+t)\, dx = \int_{a+t}^{b+t} f(x)\, dx.$$

Hint: Consider first characteristic function, simple nonnegative function, nonnegative function, and use monotone convergence theorem.

Corollary 3.3. The monotone convergence theorem also applies to decreasing sequences of functions.

Proof. Let f_n be a sequence of non-negative and increasing measurable functions that converge to f. The trick is to define $g_n = f_1 - f_n$, which is a sequence of non-negative and increasing measurable functions. By the monotone convergence theorem, we have

$$\lim_{n\to\infty} \int g_n \, d\mu = \lim_{n\to\infty} \int f_1 - f_n \, d\mu = \int f_1 \, d\mu - \int f \, d\mu.$$

This implies $g_n \to f_1 - f$ pointwise. It follows that

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu,$$

so indeed, the monotone convergence theorem applies to f_n as well. □

Corollary 3.4 (simple function). A function is simple if and only if it is of the form

$$\sum_{i=1}^{n} c_i \chi_{E_i}.$$

Moreover, if each E_i is measurable, then $\sum_{i=1}^{n} c_i \chi_{E_i}$ is measurable.

Lemma 3.9. Suppose

$$f = \sum_{i=1}^{n} c_i \chi\left(E_i\right) \quad \text{is a simple function} \quad \text{with } E_i \text{ measurable.}$$

Then,

$$\int \sum_{i=1}^{n} c_i \chi(E_i) = \sum_{i=1}^{n} c_i \mu(E_i).$$

Example 3.11 (Axler p. 87 Question 19). Show that if $(X, \mathscr{S}, \mu)$ is a measure space and $f : X \to [0, \infty)$ is $\mathscr{S}$-measurable, then

$$\mu(X) \inf_X f \leq \int f \, d\mu \leq \mu(X) \sup_X f.$$

Solution. Again, we turn to the definition of the Lebesgue integral. First, it is clear that

$$\inf_X f \leq f \quad \text{for all functions } f : X \to [0, \infty).$$

Integrating both sides with respect to the measure μ, we have

$$\int \inf_X f \, d\mu \leq \int f \, d\mu.$$

Since $\inf_X f$ is constant, then the LHS of the inequality is merely $\inf_X f \cdot \mu(X)$. The upper bound of the integral of f with respect to μ can be deduced in a similar manner. □

Example 3.12. If a function $\varphi : \Omega \to \mathbb{R}$ is given by

$$\varphi(x) = \sum_{k=1}^{m} b_k \chi_{E_k}(x),$$

where b_k are nonnegative real numbers, and $E_k \in \mathscr{A}$, a σ-algebra.

(a) Show that φ is a simple function.

(b) Suppose $(\Omega, \mathscr{A}, \mu)$ is a measure space. Show that

$$\int \varphi d\mu = \sum_{k=1}^{m} b_k \mu\left(E_k\right)$$

Solution.

(a) Recall that a function $f:\Omega\to\mathbb{R}$ is simple if there exist disjoint sets $E_1,\ldots,E_n$ such that $\Omega=\bigcup E_i$ and on each E_i, f is a constant. Hence, in this context, it suffices to prove that the range of φ is finite. Since φ depends on which sets E_k contain x, then the aforementioned linear combination takes on combinations of the b_k's depending on which sets E_k overlap. Since there are finitely many E_k's, the possible values that $\varphi(x)$ can take form a finite set of $\mathbb{R}_{\geq 0}$.

(b) We have

$$\begin{aligned}\int \varphi\, d\mu &= \int \sum_{k=1}^{m} b_k\chi_{E_k}(x)\, d\mu\\ &= \sum_{k=1}^{m} b_k\left(\int \chi_{E_k}(x)\, d\mu\right)\\ &= \sum_{k=1}^{m} b_k\int_{\chi_{E_k}} \mathbf{1}\, d\mu\end{aligned}$$

and the result follows by the definition of the characteristic function χ_E since $\chi_E(x)=1$ for all $x\in E$ and 0 elsewhere. □

Example 3.13. Let $M(\Omega,\mathscr{A})$ denote the collection of measurable functions. Show that if $\varphi,\psi\in M(\Omega,\mathscr{A})$ are simple functions and $c\in\mathbb{R}$, then $c\varphi$, $|\varphi|,\varphi+\psi,\varphi\psi$ and $\max\{\varphi,\psi\}$ are all simple functions.

Solution. Since φ and ψ are simple, then there exist $a_1,\ldots,a_m,b_1,\ldots,b_n\in\mathbb{R}$ and disjoint sets $E_1,\ldots,E_m$ and $F_1,\ldots,F_n$ (where $\Omega=\bigsqcup E_i=\bigsqcup F_j$) such that

$$\varphi(x)=\sum_{i=1}^{m} a_i\chi_{E_i}(x)\quad\text{and}\quad \psi(x)=\sum_{j=1}^{n} b_j\chi_{F_j}(x).$$

We first prove that $c\varphi$ is simple. To see why,

$$c\phi(x)=\sum_{i=1}^{m} ca_i\chi_{E_i}(x).$$

Also, $|\varphi|$ is measurable as

$$|\varphi(x)|=\sum_{i=1}^{m} a_i'\chi_{E_i}(x),\quad \text{where } a_i'=|a_i|.$$

Also, $\varphi+\psi$ is measurable as

$$\varphi+\psi=\varphi(x)=\sum_{i=1}^{m} a_i\chi_{E_i}(x)+\sum_{j=1}^{n} b_j\chi_{F_j}(x)=\sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij}\chi_{G_{ij}}(x).$$

Here, $d_{ij}=a_i+b_j$ and $G_{ij}=E_i\cap F_j$. One checks that

$$G_{ij}\text{ are pairwise disjoint}\quad\text{since}\quad \bigsqcup_{i=1}^{m}\bigsqcup_{j=1}^{n} G_{ij}=\bigsqcup_{i=1}^{m} E_i=\bigsqcup_{j=1}^{n} F_j=\Omega.$$

We then prove that $\varphi\psi$ is simple. To see why, we have

$$\varphi(x)\psi(x)=\left(\sum_{i=1}^{m} a_i\chi_{E_i}(x)\right)\left(\sum_{j=1}^{n} b_j\chi_{F_j}(x)\right)=\sum_{i=1}^{m}\sum_{j=1}^{n} a_ib_j\chi_{G_ij}(x).$$

Also,

$$\max\{\varphi(x),\psi(x)\}=\sum_{i=1}^{m}\sum_{j=1}^{n}\max\{a_i,b_j\}\chi_{G_{ij}}(x).$$

which establishes $\max\{\varphi,\psi\}$ is measurable. □

Example 3.14. Let $\Omega=\mathbb{N}$, let $\mathscr{A}$ be all subsets of $\mathbb{N}$, and let μ be the counting measure on $\mathscr{A}$ (i.e. $\mu(\mathscr{A})$ denotes the cardinality of $\mathscr{A}$). Show that if f is a non-negative function on $\mathbb{N}$, then f is a non-negative measurable function and

$$\int f d\mu=\sum_{n=0}^{\infty}f(n).$$

Solution. For the first part, we need to prove for every Borel set $B\subseteq[0,\infty)$, we have $f^{-1}(B)\in\mathscr{A}$. This is clear because $f^{-1}(B)\subseteq\mathbb{N}\in\mathscr{A}$.

As for the second part, let

$$f_n:\mathbb{N}\to\mathbb{R}\quad\text{such that}\quad f_n(k)=\begin{cases}f(k) & \text{if } 0\le k\le n;\\ 0 & \text{if } k>n.\end{cases}$$

Then, $f_n\to f$ pointwise as n tends to infinity. Also, since f is an increasing sequence of non-negative functions, by the monotone convergence theorem, we have

$$\lim_{n\to\infty}\int_{\mathbb{N}}f_n\,d\mu=\int_{\mathbb{N}}f\,d\mu.$$

Note that

$$\mathbb{N}=\left(\bigsqcup_{k=0}^{n}\{k\}\right)\cup\{n+1,n+2,\ldots\}$$

and all these sets are measurable. Hence,

$$\begin{aligned}\int_{\mathbb{N}}f\,d\mu&=\sum_{k=0}^{n}\int_{\{k\}}f_n\,d\mu+\int_{\{n+1,n+2,\ldots\}}f_n\,d\mu\\&=(f(0)+f(1)+\ldots+f(n))+0=\sum_{n=0}^{\infty}f(n)\end{aligned}$$

and the result follows. □

Lemma 3.10. Suppose A_n is a non-decreasing sequence of measurable sets and

$$A=\bigcup_n A_n.$$

Suppose ϕ is a measurable simple function. Then,

$$\int_{\Omega}\phi\cdot\chi_A\,d\mu=\int_A\phi\,d\mu=\lim_{n\to\infty}\int_{A_n}\phi\,d\mu$$

Lemma 3.11. Suppose $0 \le f \le g$ are measurable functions. Then,

$$0 \le \int f \, d\mu \le \int g \, d\mu.$$

Then, the following hold:

(i)

$$0 \le \int f \, d\mu \le \int g \, d\mu$$

(ii) If $E \subseteq F$ are measurable, then

$$\int_E f \, d\mu \le \int_F f \, d\mu$$

Proposition 3.2. Suppose f, g are integrable functions and $\alpha, \beta \in \mathbb{R}$. Then,

$$\alpha f + \beta g \text{ is measurable} \quad \text{and} \quad \int (\alpha f + \beta g) \, d\mu = \alpha \int f \, d\mu + \beta \int g \, d\mu.$$

Example 3.15. Suppose f and g are integrable functions on a measure space $(\Omega, \mathscr{A}, \mu)$. Show that $f+g$ is also integrable.

Solution. We have

$$\begin{aligned} \int_\Omega |f+g| \, d\mu &\le \int_\Omega |f| \, d\mu + \int_\Omega |g| \, d\mu \quad \text{by triangle inequality} \\ &= M + N \quad \text{since } f \text{ and } g \text{ are integrable.} \end{aligned}$$

Since the integral of $|f+g|$ is finite, then $f+g$ is integrable. □

We have been using the following notation a few times but we will now formally define it.

Definition 3.11. Let $M^+(\Omega, \mathscr{A})$ denote the collection of non-negative measurable functions.

Lemma 3.12. Suppose $f \in M^+(\Omega, \mathscr{A}, \mu)$. Then,

$$\int f \, d\mu = 0 \quad \text{if and only if} \quad f = 0 \text{ almost everywhere.}$$

Proof. For the forward direction, it suffices to show that if $E = \{x \in \mathbb{R} : f(x) > 0\}$ is such that $\mu(E) > 0$, then

$$\int f \, d\mu > 0.$$

Our goal is to find $c\chi_{E'}$ such that

$$c\chi_{E'} \le f \quad \text{and} \quad \int c\chi_{E'} \, d\mu > c.$$

Let

$$E = \bigcup E_n \quad \text{where} \quad E_n = \left\{ x \in \mathbb{R} : f(x) > \frac{1}{n} \right\}.$$

By countable additivity, $\mu(E_n) > 0$ for some E_n. Thus,

$$\int f \, d\mu \geq \int \frac{1}{n} \chi_{E_n} \, d\mu = \frac{1}{n} \mu(E_n) > 0.$$

As for the reverse direction, it suffices to prove that for any simple function ϕ such that $\phi \leq f$, we have

$$\int \phi \, d\mu.$$

Since $\phi \leq f$, then $\{x \in \mathbb{R} : \phi(x) > 0\} \subseteq \{x : f(x) > 0\}$. Hence, the result follows. $\square$

Lemma 3.13. A measurable function f is integrable if and only if $|f|$ is measurable.

Proof. We first prove the reverse direction. Say $|f|$ is integrable. Then,

$$f^+ \leq |f| \quad \text{and} \quad f^- \leq |f|.$$

Recall that

$$\begin{aligned} \int f^+ &= \sup \left\{ \int \phi \, d\mu : \phi \leq f^+ \right\} \\ &\leq \sup \left\{ \int \psi \, d\mu : \psi \leq |f| \right\} = \int f \, d\mu \end{aligned}$$

Hence, the reverse direction follows.

As for the reverse direction, recall that $|f| = f^+ + f^-$. When

$$\int f^+ \, d\mu \quad \text{and} \quad \int f^- \, d\mu \quad \text{are finite,}$$

we have

$$\int |f| \, d\mu = \int f^+ \, d\mu + \int f^- \, d\mu \quad \text{which is finite.}$$

$\square$

Corollary 3.5 (triangle inequality for integrals)**.** We have

$$\left| \int f \, d\mu \right| \leq \int |f| \, d\mu$$

Proof. We have

$$\begin{aligned}|f\,d\mu| &= \left|\int f^+\,d\mu - \int f^-\,d\mu\right| \\ &\le \left|\int f^+\,d\mu\right| + \left|\int f^-\,d\mu\right| \\ &= \int f^+\,d\mu + \int f^-\,d\mu \\ &= \int \left(f^+ + f^-\right)\,d\mu = \int |f|\,d\mu\end{aligned}$$

□

Corollary 3.6. Suppose f and g are measurable, $|f| \le |g|$, and g is integrable. Then, f is measurable.

Proposition 3.3. Suppose f and g are integrable and $\alpha, \beta \in \mathbb{R}$. Then, $\alpha f + \beta g$ is integrable. Thus, the collection of integrable functions forms a vector subspace of the collection of measurable functions.

Example 3.16 (MA4262 AY24/25 Tutorial 6)**.** Let $L(\Omega, \mathscr{A}, \mu)$ be the set of Lebesgue integrable functions on the measure space $(\Omega, \mathscr{A}, \mu)$. If $f \in L(\Omega, \mathscr{A}, \mu)$ and $a > 0$, show that the set $\{x \in \Omega : |f(x)| \ge a\}$ has finite measure.

Solution. Let he mentioned set be E_a. Then, we have

$$\int_\Omega |f|\,d\mu \ge \int_{E_a} |f|\,d\mu \ge \int_{E_a} a\,d\mu = a \cdot \mu(E_a) \quad \text{which implies} \quad \mu(E_a) \le \frac{1}{a}\int_\Omega |f|\,d\mu$$

Since $|f|$ is Lebesgue integrable, then it follows that $\mu(E_a)$ is finite. □

Theorem 3.5 (Fatou's lemma)**.** Suppose f_n is a sequence of non-negative measurable functions. Then,

$$\int \liminf f_n\,d\mu \le \liminf \int f_n\,d\mu.$$

Example 3.17. Let h be in $M^+(\Omega, \mathscr{A})$, and suppose that $\int h d\mu < \infty$. If (f_n) is a sequence of in $M(\Omega, \mathscr{A})$ and if $-h \le f_n$, then

$$\int (\liminf f_n)\,d\mu \le \liminf \int f_n d\mu.$$

Hint: Use Fatou's lemma.

Solution. Applying Fatou's lemma to $f_n + h$, we have

$$\int \liminf h\,d\mu + \int \liminf f_n\,d\mu \le \int \liminf (h + f_n)\,d\mu \le \liminf \int h + f_n\,d\mu.$$

Hence,

$$\int \liminf h\,d\mu + \int \liminf f_n\,d\mu \le \liminf \int h\,d\mu + \liminf \int f_n\,d\mu.$$

The result follows. □

Example 3.18 (Axler p. 99 Question 1)**.** Give an example of a sequence $f_1, f_2, \ldots$ of functions from $\mathbb{Z}^+$ to $[0,\infty)$ such that

$$\lim_{k\to\infty} f_k(m) = 0 \quad \text{for every } m \in \mathbb{Z}^+$$

but

$$\lim_{k\to\infty} \int f_k\,d\mu = 1, \quad \text{where } \mu \text{ is the counting measure on } \mathbb{Z}^+.$$

Solution. Consider

$$f_k(m) = \begin{cases} 1/m & \text{if } m \le k; \\ 0 & \text{otherwise.} \end{cases}$$

Then, the first limit is clearly 0 since $f_k(m) = 0$ if $m > k$. Also,

$$\lim_{k\to\infty} \int f_k\,d\mu = \sum_{m=1}^{\infty} f_k(m) = \sum_{m=1}^{k} \frac{1}{k} = \frac{1}{k}\cdot k = 1.$$

As such, our suggested sequence of functions satisfies the claim. □

3.3. *Dominated Convergence Theorem*

Theorem 3.6 (dominated convergence theorem)**.** Suppose f_n is a sequence of integrable functions, $f_n \to f$ almost everywhere, and f is measurable. Suppose there exists an integrable g such that

$$|f_n| \le |g| \quad \text{for all } n.$$

Then, f is integrable and

$$\int f\,d\mu = \lim_{n\to\infty} \int f_n\,d\mu.$$

Example 3.19. We consider an example where the dominated convergence theorem is not applicable.

$$f_n(x) = \begin{cases} n/2 & \text{for } -1/n < x < /1n; \\ 0 & \text{otherwise.} \end{cases}$$

Then, $f_n \to 0$ pointwise but

$$\int \lim f_n\,d\mu \ne \lim_{n\to\infty} \int f_n\,d\mu$$

since the LHS is 0 but the RHS is 1.

We now prove the dominated convergence theorem.

Proof. Suppose $f_n+|g| \to f+|g|$ pointwise, where $f_n+|g|, f+|g| \geq 0$. By Fatou's lemma, we have

$$\begin{aligned}\int f+|g|\ d\mu &\leq \liminf \int f_n+|g|\ d\mu \\ &\leq \int 2|g|\ d\mu\end{aligned}$$

which implies

$$\begin{aligned}\int f\ d\mu + \int |g|\ d\mu &\leq \int |g|\ d\mu + \liminf \int f_n\ d\mu \\ \int f\ d\mu &\leq \liminf \int f_n\ d\mu\end{aligned}$$

On the other hand, we have $|g|-f_n \to |g|-f$, where $|g|-f_n, |g|-f \geq 0$. By Fatou's lemma,

$$\begin{aligned}\int |g|-f\ d\mu &\leq \liminf \int |g|-f_n\ d\mu \\ \int f-|g|\ d\mu &\geq \limsup \int f_n-|g|\ d\mu \\ \int f\ d\mu - \int |g|\ d\mu &\geq \limsup \int f_n\ d\mu - \int |g|\ d\mu \\ \int f\ d\mu &\geq \limsup \int f_n\ d\mu\end{aligned}$$

Since we showed that

$$\limsup \int f_n\ d\mu \leq \int f\ d\mu \leq \liminf \int f_n\ d\mu$$

Thus,

$$\int f\ d\mu = \liminf \int f_n\ d\mu = \limsup \int f_n\ d\mu = \lim \int f_n\ d\mu$$

and the result follows. □

Example 3.20 (MA4262 AY14/15 Sem 1 Tutorial 7)**.** If f is an integrable function on a measurable set E and

$$E = \bigcup_{i=1}^{\infty} E_i \quad \text{such that } E_i \text{ is measurable} \quad \text{and} \quad E_i \cap E_j = \emptyset \text{ for distinct } i,j,$$

prove that

$$\sum_{i=1}^{\infty} \int_{E_i} f = \int_E f.$$

Solution. The fact that $E_i \cap E_j = \emptyset$ for distinct i,j means that E is a disjoint union of the E_i's. Hence,

$$\int_E f = \int_{E_1} f + \int_{E_2} f + \ldots = \lim_{n\to\infty} \sum_{i=1}^{n} \int_{E_i} f$$

and the result follows. Note that we used the countable additivity of the Lebesgue integral. □

Example 3.21 (MA4262 AY24/25 Sem 1 Tutorial 6)**.** Suppose (f_n) is a sequence in $L(\Omega,\mathscr{A},\mu)$ which converges uniformly on Ω to a real-valued function f.

(a) If $\mu(\Omega) < +\infty$, then

$$\int f\,d\mu = \lim \int f_n\,d\mu.$$

Hint: Use the Lebesgue dominated convergence theorem, choosing g appropriately.

(b) Show by example that if $\mu(\Omega) = +\infty$, the convergence may fail.

Solution.

(a) Since $f_n \to f$ uniformly on Ω, then for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $|f_n - f| < \varepsilon/\mu(\Omega)$, so

$$\left|\int_\Omega f_n - f\,d\mu\right| \leq \int_\Omega |f_n - f|\,d\mu \leq \int_\Omega \frac{\varepsilon}{\mu(\Omega)} \cdot \mu(\Omega)\,d\mu = \varepsilon.$$

Here, we did not use the Lebesgue dominated convergence theorem but the proof still works out.

(b) Suppose $\Omega = [0,\infty)$ and

$$f_n(x) = \begin{cases} 1/n & \text{if } 0 \leq x \leq n; \\ 0 & \text{if } x > n. \end{cases}$$

Note that $f_n \to f$ uniformly because

$$\sup \|f_n - f\| \to 0.$$

However,

$$\int f\,d\mu = \int 0\,d\mu = 0 \quad \text{but} \quad \lim_{n\to\infty} \int f_n\,d\mu = \lim_{n\to\infty} \int_0^\infty \frac{1}{n}\,d\mu = 1$$

which shows that convergence fails. □

Example 3.22 (MA4262 AY24/25 Sem 1 Tutorial 6)**.** Let $L(\Omega,\mathscr{A},\mu)$ be the set of Lebesgue measurable functions on the measure space $(\Omega,\mathscr{A},\mu)$. Let $f_n \in L(\Omega,\mathscr{A},\mu)$, and suppose that (f_n) converges to a function f. Show that if

$$\lim_{n\to\infty} \int |f_n - f|\,d\mu = 0 \quad \text{then} \quad \lim_{n\to\infty} \int |f_n|\,d\mu = \int |f|\,d\mu.$$

Solution. This is a simple proof using the formal definition of limits. □

Theorem 3.7. A function $f : [a,b] \to \mathbb{R}$ is Riemann integrable if and only if f is Lebesgue integrable and the set $\{x \in \mathbb{R} : f \text{ not continuous at } x\}$ has measure 0.

Recall that if $P = \{x_1,\ldots,x_n\}$ is a partition of an interval $[a,b]$, where $a = x_1$ and $b = x_n$, then we can define

$$U(f,P) = \sum_{i=1}^{n} M_i \Delta x_i \quad \text{and} \quad L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i.$$

Here, M_i and m_i denote the supremum and infimum of the partition respectively. We define

$$\int^{\text{UR}} f(x)\,dx = U(f) = \inf_P U(f,P) \quad \text{and} \int^{\text{LR}} f(x)\,dx = L(f) = \sup_P L(f,P).$$

We say that f is Riemann integrable if and only if

$$\int^{\text{UR}} f(x)\,dx = \int^{\text{LR}} f(x)\,dx.$$

Example 3.23 (MA4262 AY24/25 Sem 1 Tutorial 5)**.** Prove or disprove: If $f \in M^+(\mathbb{R}, \mathscr{L}, \lambda)$ is integrable and so is $\int f^{2024}\,d\lambda$. (Here, f^{2024} is given by $f^{2024}(x) = (f(x))^{2024}$).

Solution. The statement is false. Consider $f(x) = 1/\sqrt{x}$ on $[0,1]$, which is integrable. However, $f^2 = 1/x$ is not integrable on $[0,1]$. Consequently, f^{2024} is not integrable on $[0,1]$. □

Lemma 3.14. Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then,

$$\int^{\text{UR}} f(x)\,dx - \int^{\text{LR}} f(x)\,dx = \int \omega_f(x)\,d\mu.$$

Here, ω is the oscillation function defined earlier.

Remark 3.3. Actually, the monotone convergence theorem, dominated convergence theorem, and Fatou's lemma appear to be *equivalent*.

Example 3.24 (MA4262 AY24/25 Sem 1 Tutorial 5)**.** If $(\Omega, \mathscr{A}, \mu)$ is a finite measure space, (that is, $\mu(\Omega) < +\infty$,) and if (f_n) is a real-valued sequence in $M^+(\Omega, \mathscr{A})$ which converges uniformly to a function $f \in M^+(\Omega, \mathscr{A})$. Show that

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu.$$

Hint: Consider two cases $\int f d\mu < \infty$ and $\int f d\mu = \infty$. In the first case, use dominated convergence theorem.

Solution. Suppose $f_n \to f$ pointwise and say f_n is dominated by some function g, i.e. for every $n \in \mathbb{N}$, there exists g such that $|f_n| \le |g|$. We consider the first case, i.e. $f_n \to f$ uniformly such that

$$\int f\,d\mu < \infty.$$

Then, because for every $\varepsilon > 0$, we have $|f_n - f| < \varepsilon$, then we can set $g = f + \varepsilon$, so by the dominated convergence theorem,

$$\int f + \varepsilon\,d\mu > \lim_{n\to\infty} \int f_n\,d\mu$$

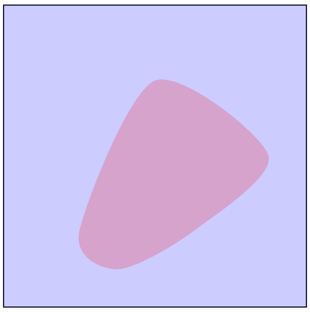
Since ε can be made arbitrarily small, then the result follows. On the other hand, for the case when

$$\int f\,d\mu = \infty,$$

we have $|f_n - f| < 1$, so the result follows trivially. □

3.4. *Fubini-Tonelli Theorem*

We begin with some recap from MA2104.



Use the above picture to think of the following concept:

$$\int_{\mathbb{R}^2} f(x,y)\,dxdy = \int \left(\int f(x,y)\,dy \right) dx = \int \left(\int f(x,y)\,dx \right) dy$$

Recall that if

$$(\Omega_1, \mathscr{A}_1, \mu_1) \text{ and } (\Omega_2, \mathscr{A}_2, \mu_2) \quad \text{are measure spaces,}$$

on $\Omega_1 \times \Omega_2$, we can introduce the product σ-algebra $\mathscr{A}_1 \otimes \mathscr{A}_2$, which is the smallest σ-algebra generated by $A_1 \times A_2$ with $A_1 \in \mathscr{A}_1$ and $A_2 \in \mathscr{A}_2$. The product measure $\mu = \mu_1 \otimes \mu_2$ is a measure such that $\mu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2)$.

Also, Ω is σ-finite if

$$\Omega = \bigsqcup \Omega_n \quad \text{and} \quad \mu(\Omega_n) < \infty.$$

If Ω_1 and Ω_2 are σ-finite, then the product measure of μ_1 and μ_2 exists uniquely by Carathéodory's extension theorem. When $\mathscr{A}$ is the product $\mathscr{A}_1 \otimes \mathscr{A}_2$ and μ is the product measure $\mu_1 \otimes \mu_2$, we say that $(\Omega, \mathscr{A}, \mu)$ is the product measure space of $(\Omega_1, \mathscr{A}_1, \mu_1)$ and $(\Omega_2, \mathscr{A}_2, \mu_2)$.

Theorem 3.8 (Fubini's theorem)**.** Suppose $(\Omega_1, \mathscr{A}_1, \mu_1)$ and $(\Omega_2, \mathscr{A}_2, \mu_2)$ are σ-finite measure spaces and $(\Omega, \mathscr{A}, \mu)$ is their product. Let f be integrable on $(\Omega, \mathscr{A}, \mu)$. Then, for $a \in \Omega_1, b \in \Omega_2$ almost everywhere, the functions

$$f_a : \Omega_2 \to \mathbb{R}, y \mapsto f(a,y) \quad \text{and} \quad f_b : \Omega_1 \to \mathbb{R}, x \mapsto f(x,b) \quad \text{are integrable.}$$

Moreover,

$$\int_\Omega f\,d\mu = \int_{\Omega_1} \left(\int_{\Omega_2} f_a\,d\mu_2 \right) d\mu_1 = \int_{\Omega_2} \left(\int_{\Omega_1} f_b\,d\mu_1 \right) d\mu_2.$$

Theorem 3.9 (Tonelli's theorem). Let $f \geq 0$ be a measurable function on $(\Omega, \mathscr{A}, \mu)$, where $(\Omega_1, \mathscr{A}_1, \mu_1)$ and $(\Omega_2, \mathscr{A}_2, \mu_2)$ are σ-finite measure spaces and $(\Omega, \mathscr{A}, \mu)$ is their product. Then,

$$\int_\Omega f\,d\mu = \int_{\Omega_1} (f_a\,d\mu_2)\;d\mu_1 = \int_{\Omega_2} (f_b\,d\mu_1)\;d\mu_2.$$

Remark 3.4. We were a bit not rigorous when talking about

$$\int_{\Omega_1} \left(\int_{\Omega_2} f_a\,d\mu_2\right)\,d\mu_1 \quad \text{as } f_a \text{ is only integrable almost everywhere.}$$

However, this can be easily remedied by defining

$$\int_{\Omega_1} \left(\int_{\Omega_2} f_a\,d\mu_2\right)\,d\mu_1 \quad \text{as} \quad \int_{\Omega_1 \setminus N_1} \left(\int_{\Omega_2} f_a\,d\mu_2\right)\,d\mu_1.$$

Here, N_1 is the set of all a where f is not intergable.

Example 3.25 (Axler p. 135 Question 1).

(a) Let λ denote Lebesgue measure on $[0,1]$. Show that

$$\int_{[0,1]} \int_{[0,1]} \frac{x^2 - y^2}{(x^2+y^2)^2}\,d\lambda(y)\,d\lambda(x) = \frac{\pi}{4}$$

and

$$\int_{[0,1]} \int_{[0,1]} \frac{x^2 - y^2}{(x^2+y^2)^2}\,d\lambda(x)\,d\lambda(y) = -\frac{\pi}{4}.$$

(b) Explain why **(a)** violates neither Tonelli's Theorem nor Fubini's Theorem.

Solution.

(a) We will only prove that the first integral indeed evaluates to $\pi/4$. Note that

$$\begin{aligned}\frac{x^2-y^2}{(x^2+y^2)^2} &= \frac{x^2+y^2}{(x^2+y^2)^2} - \frac{2y^2}{(x^2+y^2)^2} \\ &= \frac{1}{x^2+y^2} - \frac{2x^2+2y^2}{(x^2+y^2)^2} + \frac{2x^2}{(x^2+y^2)^2} \\ &= -\frac{1}{x^2+y^2} + \frac{2x^2}{(x^2+y^2)^2}\end{aligned}$$

The remaining calculations are trivial so we will not go through them.

(b) Tonelli's theorem is not violated because it is possible for the integrand $f(x,y)$ to be negative, i.e.

$$\frac{x^2-y^2}{(x^2+y^2)^2} \leq 0 \quad \text{if and only if} \quad x^2 \leq y^2.$$

Recall that Tonelli's theorem requires the integrand to be non-negative. As for Fubini's theorem, the integral is not absolutely convergent. Observe that

$$\int_0^1 \int_0^1 \left|\frac{x^2-y^2}{(x^2+y^2)^2}\right|\,dydx = \int_0^1 \int_0^x \frac{x^2-y^2}{(x^2+y^2)^2}\,dydx + \int_0^1 \int_0^y \frac{y^2-x^2}{(x^2+y^2)^2}\,dydx$$

for which one can verify that it is the sum of two divergent integrals. □

Example 3.26 (Axler p. 100 Question 14). Let λ denote Lebesgue measure on $\mathbb{R}$.

(a) Let $f(x)=1/\sqrt{x}$. Prove that

$$\int_{[0,1]} f\,d\lambda = 2.$$

(b) Let $f(x)=1/(1+x^2)$. Prove that

$$\int_{\mathbb{R}} f\,d\lambda = \pi.$$

(c) Let $f(x)=\sin x/x$. Show that

$$\int_{(0,\infty)} f\,d\lambda \text{ is not defined} \quad \text{but} \quad \lim_{t\to\infty}\int_{(0,t)} f\,d\lambda \text{ exists in } \mathbb{R}.$$

Solution. **(a)** and **(b)** are trivial as it is easy to see that

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2 \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \pi \quad \text{respectively.}$$

The aforementioned integrals are absolutely convergent so their Lebesgue integrals exists. As for **(c)**, we first need to show that $\sin x/x$ is not Lebesgue integrable on $(0,\infty)$, i.e. the function is not absolutely convergent. We shall consider

$$\int_0^\infty \left|\frac{\sin x}{x}\right|\,dx \geq \int_1^\infty \frac{|\sin x|}{x}\,dx = \sum_{n=1}^\infty \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x}\,dx \geq \sum_{n=1}^\infty \frac{1}{n\pi}\int_{n\pi}^{(n+1)\pi} |\sin x|\,dx = \frac{2}{\pi}\sum_{n=1}^\infty \frac{1}{n}$$

which diverges as this is simply the harmonic series!

On the other hand,

$$\begin{aligned}
\int_0^\infty \frac{\sin x}{x}\,dx &= \int_0^\pi \frac{\sin x}{x}\,dx + \sum_{n=1}^\infty \frac{1}{(n+1)\pi}\int_{n\pi}^{(n+1)\pi} \sin x\,dx \\
&\leq \int_0^\pi \frac{\sin x}{x}\,dx + \sum_{n=1}^\infty \frac{1}{(n+1)\pi}\int_{n\pi}^{(n+1)\pi} \sin x\,dx \\
&\leq \int_0^\pi \frac{x}{x}\,dx + \frac{2}{\pi}\sum_{n=1}^\infty \frac{(-1)^n}{n+1} \\
&= \pi + \frac{2}{\pi}(\ln 2 - 1)
\end{aligned}$$

which is finite. □

Remark 3.5. The Lebesgue measure on $\mathbb{R}\times\mathbb{R}$ is not the product measure of Lebesgue measures on $\mathbb{R}$.

4. Convergence

4.1. L^p Spaces

Definition 4.1 (L^p space)**.** Let $(\Omega, \mathscr{A}, \mu)$ be a measure space.

(i) If $1 < p < \infty$, the L^p space $L^p(\Omega, \mathscr{A}, \mu)$ on $(\Omega, \mathscr{A}, \mu)$ consists of all measurable functions f on Ω such that

$$\int |f(x)|^p \, d\mu < \infty \quad \text{or equivalently} \quad \|f\|_p = \left| \int |f(x)|^p \, d\mu \right|^{1/p} < \infty.$$

(ii) If $p = \infty$, then $L^\infty(\Omega, \mathscr{A}, \mu)$ on $(\Omega, \mathscr{A}, \mu)$ consists of measurable functions f such that there exists $M > 0$ with $f(x) \le M$ almost everywhere for $x \in \Omega$. Equivalently,

$$\|f\|_\infty = \inf \{M : \exists N \in \mathscr{A}, \mu(N) = 0 \text{ and for all } x \notin N, |f(x)| \le M\}.$$

In the above definition on the L^p space, $\|f\|_\infty$ is also often called the essential supremum of f. $\|f\|_p$ is often called the p-norm of f.

Example 4.1 (L^1 space)**.** $L^1(\Omega, \mathscr{A}, f)$ is the space of intgerable functions.

Example 4.2 (L^2 space)**.** $L^2(\Omega, \mathscr{A}, f)$ is the space of square-integrable functions, i.e.

$$\left(\int |f|^2 \, d\mu \right)^{1/2} < \infty.$$

Recall the following (not sure if it was mentioned earlier):

Proposition 4.1. The following are equivalent:

$$\int |f| \, d\mu < \infty \quad \text{if and only if} \quad |f| \text{ integrable} \quad \text{if and only if} \quad f \text{ integrable}.$$

Proposition 4.2 (L^p space is a vector space)**.** For all $1 < p \le \infty$, $L^p(\Omega, \mathscr{A}, \mu)$ is a vector space over $\mathbb{R}$ with the obvious addition and scalar multiplication.

We will not state the axioms for a set V to be considered a vector space as they should be pretty obvious. Note that the set V should be an Abelian group satisfying scalar multiplication properties.

We now prove that Proposition 4.2 holds.

Proof. Note that the space of real-valued functions forms a vector space over $\mathbb{R}$. Hence, it suffices to check that $L^p(\Omega, \mathscr{A}, \mu)$ is a vector subspace of this space, i.e. we need to check that $L^p(\Omega, \mathscr{A}, \mu)$ is closed under addition and scalar multiplication.

For $1 < p < \infty$, given that

$$\|f\|_p < \infty, \|g\|_p < \infty, \quad \text{we need to check that } \|f+g\|_p < \infty.$$

We have

$$\int |f+g|^p \, d\mu \le \int (|f|+|g|)^p \, d\mu \le \int (2\max\{|f|,|g|\})^p \, d\mu \le \int 2^p \left(|f|^p + |g|^p\right) d\mu < \infty.$$

For any $\alpha \in \mathbb{R}$, it is easy to show that $\|\alpha f\|_p < \infty$ since

$$\int |af|^p \, d\mu = \alpha^p \int |f|^p \, d\mu < \infty.$$

For $p = \infty$, given that

$$\|f\|_\infty, \|g\|_\infty < \infty, \quad \text{we need to check that } \|f+g\|_\infty < \infty.$$

For any $x \notin N_1, x \notin N_2$, $|f| < M_1$ and $|g| < M_2$ respectively. So, for $x \notin N_1 \cup N_2$, $|f+g| < M_1 + M_2$. In fact, the case when $p = \infty$ also appears in Axler p. 199 Question 1. □

Example 4.3 (MA4262 AY24/25 Sem 1 Tutorial 7). True or false?

(a) If $1 \le p < q \le \infty$ then $L^p(\Omega, \mathscr{A}, \mu) \subseteq L^q(\Omega, \mathscr{A}, \mu)$.

(b) If $1 \le p < \infty$ and $f, g \in L^p(\Omega, \mathscr{A}, \mu)$ then $f \cdot g \in L^p(\Omega, \mathscr{A}, \mu)$.

(c) If $f, g \in L^\infty(\Omega, \mathscr{A}, \mu)$ then $f \cdot g \in L^\infty(\Omega, \mathscr{A}, \mu)$.

(d) If $f \in L^p(\Omega, \mathscr{A}, \mu), 1 \le p < \infty$ and $g \in L^\infty(\Omega, \mathscr{A}, \mu)$, then $fg \in L^p(\Omega, \mathscr{A}, \mu)$ and $\|fg\|_p \le \|f\|_p \|g\|_\infty$.

(e) There is a function $f \in L^2([0, \pi])$ satisfying both

$$\int_{[0,\pi]} [f(x) - \sin x]^2 dx \le \frac{4}{9} \quad \text{and} \quad \int_{[0,\pi]} [f(x) - \cos x]^2 dx \le \frac{1}{9}.$$

Solution.

(a) False but the reverse inclusion holds.

(b) False. We have

$$\left(\int |f|^p \, d\mu\right)^{1/p} \quad \text{and} \quad \left(\int |g|^p \, d\mu\right)^{1/p} \quad \text{being finite.}$$

Then,

$$\left(\int |fg|^p \, d\mu\right)^{1/p} = \left(\int |f|^p |g|^p \, d\mu\right)^{1/p}.$$

However, we cannot create a nice inequality. A counterexample to the statement would be to consider $f(x) = g(x) = 1/\sqrt{x}$, where $\Omega = [0,1]$. f and g are integrable but fg is not.

(c) True because the product of bounded functions is also bounded. To see why, note that $\|f\|_\infty$ and $\|g\|_\infty$ are bounded, and subsequently, we use the fact that $\|fg\|_\infty \le \|f\|_\infty \|g\|_\infty$.

(d) True. We are given

$$\left(\int |f|^p \, d\mu\right)^{1/p} \text{ being finite} \quad \text{and} \quad \|g\| \le M \text{ for some } M \ge 0.$$

So,

$$\left(\int |fg|^p \, d\mu\right)^{1/p} = \left(\int |f|^p |g|^p \, d\mu\right)^{1/p} \le M \left(\int |f|^p \, d\mu\right)^{1/p}$$

and the assertion follows.

(e) Suppose such a function exists. By the triangle inequality, we have

$$|\sin x - \cos x| \le |f(x) - \sin x| + |f(x) - \cos x|.$$

Squaring both sides and integrating from 0 to π, we obtain

$$\pi \le \frac{4}{9} + \frac{1}{9} + 2\int_0^\pi |f(x) - \sin x|\,|f(x) - \cos x|\; dx.$$

We can further bound this using the Cauchy-Schwarz inequality. Note that

$$\int_0^\pi |f(x) - \sin x|\,|f(x) - \cos x|\; dx \le \left(\frac{4}{9} \cdot \frac{1}{9}\right)^2$$

so $\pi \le 4/9 + 1/9 + 2(4/81)^2 \le 1$, which is a contradiction. Hence, no such function f can exist. □

Definition 4.2 (norm and seminorm). Let V be vector space. A norm $\|\cdot\|$ on V is a map $\|\cdot\| : V \to \mathbb{R}$ such that the following are satisfied:

(1) Non-negativity: $\|v\| \ge 0$ for all $v \in V$

(2) Positive-definiteness: $\|v\| = 0$ if and only if $v = 0$

(3) Triangle inequality: $\|u+v\| \le \|u\| + \|v\|$ for any $u, v \in V$

(4) $\|\alpha v\| = \alpha \|v\|$

If we relax **(2)**, then $\|\cdot\|$ is called a seminorm on V. Also, each norm induces a metric $d(u,v) = \|u - v\|$.

Definition 4.3 (inner product). A real-valued inner product $\langle \cdot, \cdot \rangle$ on V is a map $V \times V \to \mathbb{R}$ satisfying the following properties:

(1) $\langle v, v \rangle \ge 0$

(2) $\langle u_1 + u_2, v \rangle = \langle u_1, v \rangle + \langle u_2, v \rangle$

(3) $\langle u, v_1 + v_2 \rangle = \langle u, v_1 \rangle + \langle u, v_2 \rangle$

(4) $\langle \alpha u, v \rangle = \alpha \langle u, v \rangle = \langle u, \alpha v \rangle$

(5) $\langle u, v \rangle = \langle v, u \rangle$

Note that each inner product gives us a seminorm $\|f\| = \langle f, f \rangle^{1/2}$. Also, not all norms can be obtained by an inner product. Need to use the parallelogram law

$$\|f+g\|^2 + \|f-g\|^2 = 2\|f\|^2 + 2\|g\|^2.$$

We will later see that $\|\cdot\|_p$ is a seminorm for $1 \le p \le \infty$ and $\|\cdot\|_2$ is a seminorm given by a semi inner product.

Example 4.4. $|\cdot|$ is a norm on $\mathbb{R}$.

Example 4.5. On $\mathbb{R}^n$,

$$\|(a_1, \ldots, a_n)\|_\infty = \sup_i |a_i|.$$

Example 4.6. On $\mathbb{R}^n$,

$$\|(a_1,\ldots,a_n)\|_2 = \sqrt{a_1^2+\ldots+a_n^2}.$$

Example 4.7. Take the space of functions with $[a,b] \to \mathbb{R}$. Then,

$$\|f\| = \sup\left\{f'(x) : x \in (a,b)\right\}$$

Theorem 4.1 (Young's inequality). Let $p,q \in \mathbb{R}$ with $1 < p,q < \infty$ and $1/p + 1/q = 1$. Then, for all $a,b \in \mathbb{R}$, we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof. Use the idea of convex functions. □

Corollary 4.1 (AM-GM inequality). For any $a,b \in \mathbb{R}$, we have

$$ab \leq \frac{a^2+b^2}{2}.$$

Proof. Set $p = q = 2$ in Young's inequality. □

Theorem 4.2 (Hölder's inequality). Suppose $p,q \in \mathbb{R}_{>0}$ with $1 \leq p,q \leq \infty$ and $1/p + 1/q = 1$. Say $f \in L^p(\Omega, \mathscr{A}, \mu)$ and $g \in L^q(\Omega, \mathscr{A}, \mu)$. Then,

$$\|f \cdot g\|_1 \leq \|f\|_p \cdot \|g\|_q.$$

In particular, $f \cdot g$ is integrable.

Equality holds if and only if

$$\alpha\,|f|^p = \beta\,|g|^q \text{ for } x \in \Omega \text{ almost everywhere} \quad \text{for some } \alpha,\beta \in \mathbb{R}.$$

Proof. When $p = 1, q = \infty$ or $q = 1, p = \infty$, we leave the proofs as exercises. As such, we shall focus on $1 < p < q < \infty$. If $\|f\|_p = 0$ or $\|g\|_q = 0$, then

$$\int |f|^p \, d\mu = 0 \text{ and the result follows.}$$

We shall assume that $\|f\|_p, \|g\|_q > 0$. Set

$$A = \frac{|f|}{\|f\|_p} \quad \text{and} \quad B = \frac{|g|}{\|g\|_q}.$$

By Young's inequality, we have

$$\frac{|fg|}{\|f\|_p \|g\|_q} \leq \frac{|f|^p}{p\left(\|f\|_p\right)^p} + \frac{|g|^q}{q\left(\|f\|_q\right)^q}.$$

Integrating both sides yields

$$\frac{1}{\|f\|_p\|g\|_q}\int |fg|\ d\mu \leq \frac{1}{p\left(\|f\|_p\right)^p}\int |f|^p\ d\mu + \frac{1}{q\left(\|g\|_q\right)^q}\int |g|^q\ d\mu$$

The result follows from here. □

Corollary 4.2 (Cauchy-Schwarz inequality)**.** Suppose $f,g\in L^2(\Omega,\mathscr{A},\mu)$, we have

$$\|f\|_2\|g\|_2 \geq \|fg\|_1.$$

Example 4.8. Suppose $\Omega=\{1,2,3\}$, $f=(a_1,a_2,a_3), g=(b_1,b_2,b_3)$. Then,

$$\sqrt{a_1^2+a_2^2+a_3^2}\sqrt{b_1^2+b_2^2+b_3^2} \geq |a_1b_1+a_2b_2+a_3b_3|.$$

Theorem 4.3 (Minkowski's inequality)**.** Suppose $f,g\in L^p=L^p(\Omega,\mathscr{A},\mu)$ with $1\leq p\leq\infty$. Then,

$$\|f+g\|_p \leq \|f\|_p+\|g\|_p.$$

In other words, the p-norm is indeed a norm.

Proof. We leave the proofs of $p=\infty$ and $p=1$ as simple exercises. In fact, the case when $p=1$ follows from the triangle inequality (Corollary 3.5), i.e.

$$\int |f+g|\ d\mu \leq \int |f|\ d\mu + \int |g|\ d\mu.$$

So, we assume that $1<p<\infty$. Then,

$$\begin{aligned}
|f+g|^p &= |f+g|\,|f+g|^{p-1}\\
&\leq (|f|+|g|)\,|f+g|^{p-1}\\
&= |f|\,|f+g|^{p-1}+|g|\,|f+g|^{p-1}
\end{aligned}$$

By applying Hölder's inequality to the integral of each term, we have something like

$$\int |f||f+g|^{p-1}\ d\mu \leq \|f\|_p \left\||f+g|^{p-1}\right\|_q \quad \text{where } 1/p+1/q=1.$$

We will not fill in the remaining details. □

Corollary 4.3. For $1\leq p\leq\infty$, $\|\cdot\|_p$ is a seminorm. Moreover, with

$$\langle f,g\rangle = \frac{\|f+g\|_2^2-\|f\|_2^2-\|g\|_2^2}{2},$$

$\langle\cdot,\cdot\rangle$ is a semi inner product. Furthermore,

$$\|f\|_p=0 \quad \text{if and only if} \quad f=0 \text{ almost everywhere.}$$

Proof. Verifying that $\|\cdot\|_p$ is a seminorm is quite simple. Note that $\|\alpha f\|_p = |\alpha|\,\|f\|_p$ holds because

$$\left(\int |\alpha f|^p \, d\mu\right)^{1/p} = |\alpha| \left(\int |f|^p \, d\mu\right)^{1/p}.$$

Also, $\|0\|_p = 0$. By Minkowski's inequality, it follows that $\|f+g\|_p \le \|f\| + \|g\|_p$.

We will not prove that $\langle \cdot, \cdot \rangle$ is a semi inner product. Just recall the properties and manually verify that they are satisfied, namely non-negativity, linearity in the first argument, conjugate homogeneity, and the Cauchy-Schwarz inequality.

As for the last statement, we see that $f = 0$ almost everywhere if and only if $|f| = 0$ a.e. if and only if $|f|^p = 0$ a.e. if and only if

$$\int |f|^p \, d\mu = 0$$

a.e. so by this chain of implications, the result follows. □

Example 4.9. Let $\Omega = \{1,2\}$, $f = (a_1, a_2), g = (b_1, b_2)$. Then, $f + g = (a_1 + b_1, a_2 + b_2)$. So,

$$\|f\|_2 = \sqrt{a_1^2 + a_2^2} \quad \text{and} \quad \|g\|_2 = \sqrt{b_1^2 + b_2^2}.$$

Then,

$$\|f+g\| = \sqrt{(a_1+b_1)^2 + (a_2+b_2)^2} = \sqrt{a_1^2 + b_1^2 + a_2^2 + b_2^2 + 2a_1b_1 + 2a_2b_2}.$$

Hence,

$$\frac{\|f+g\|_2^2 - \|f\|_2^2 - \|g\|_2^2}{2} = a_1b_1 + a_2b_2.$$

Example 4.10 (MA4262 AY24/25 Sem 1 Tutorial 7)**.** Let $1 \le p < \infty$. Show that if $f \in L^p(\Omega, \mathscr{A}, \mu)$ and $\varepsilon > 0$, then there exists a simple function $\varphi \in M(\Omega, \mathscr{A})$ such that $\|f - \varphi\|_p < \varepsilon$.

Solution. Since $f \in L^p(\Omega, \mathscr{A}, \mu)$, then

$$\left(\int |f|^p \, d\mu\right)^{1/p} \text{ is finite} \quad \text{so} \quad \int |f|^p \, d\mu \text{ is finite.}$$

We first wish to bound f, by a measurable function f_M, where $M > 0$. Define

$$f_M(x) = \begin{cases} f(x) & \text{if } |f| \le M; \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$\|f - f_M\|_p^p = \int_\Omega |f - f_M|^p \, d\mu = \int_{\{x \in \Omega : |f| > M\}} |f|^p \, d\mu.$$

We can choose $M > 0$ sufficiently large such that $\|f - f_M\|_p < \varepsilon/2$ so

$$\int_{\{x\in\Omega:|f|>M\}} |f|^p \, d\mu < \left(\frac{\varepsilon}{2}\right)^p.$$

For the simple function φ, we can define it to be M, where $f_M = M$. Hence,

$$|f_M - \varphi|^p \le \left(\frac{M}{N}\right)^p \quad \text{where } N \text{ is the number of partitions.}$$

The subsequent argument becomes straightforward. Just choose N sufficiently large subjected to a certain condition and apply the triangle inequality to $\|f - \varphi\|_p$. □

Definition 4.4 (Lebesgue space)**.** If $1 \le p \le \infty$, define the Lebesgue space $\widehat{L}^p(\Omega, \mathscr{A}, \mu)$ to be the quotient space of $L^p(\Omega, \mathscr{A}, \mu)$ with respect to the almost everywhere equivalence relation, i.e.

$$f \sim g \quad \text{if} \quad f = g \text{ almost everywhere.}$$

On $\widehat{L}^p$, we define addition, scalar multiplication and p-norm via the following:

$$[f] + [g] = [f+g] \quad \text{and} \quad \alpha[f] = [\alpha f] \quad \text{and} \quad \|[f]\|_p = \|f\|_p$$

Theorem 4.4. For $1 \le p \le \infty$, the Lebesgue space $\widehat{L}^p$ with the defined addition, scalar multiplication, and p-norm forms a normed vector space.

We see that when $p = 2$, there is an obvious way to define an inner product.

Example 4.11 (MA4262 AY24/25 Sem 1 Tutorial 7)**.**

(a) For $\alpha \in (0,1)$ and $t \in (0,\infty)$, by taking derivative, show that

$$t^\alpha \le \alpha t + (1-\alpha)$$

with equality happens if and only if $t = 1$.

(b) For $A, B > 0$ and $p, q \in (1,\infty)$ with $1/p + 1/q = 1$ set $t = A^p/B^q$ and $\alpha = 1/p$ to deduce the weighted AM-GM inequality

$$AB \le \frac{A^p}{p} + \frac{B^q}{q}$$

with equality happens if and only if $A^p = B^q$.

(c) For $a_1, a_2, b_1, b_2 > 0$ and $p, q \in (1,\infty)$ with $1/p + 1/q = 1$, use **(b)** and following the proof of Holder's inequality to show that

$$(a_1b_1 + a_2b_2) \le (a_1^p + a_2^p)^{1/p}(b_1^q + b_2^q)^{1/q}$$

with equality holds when (a_1^p, a_2^p) is a constant multiple of (b_1^q, b_2^q)

(d) For $c_1, c_2, d_1, d_2 > 0$ and $p \in (1, \infty)$, use **(c)** and following the proof of Minkowski's inequality to show that

$$[(c_1+d_1)^p + (c_2+d_2)^p]^{1/p} \le (c_1^p + c_2^p)^{1/p} + (d_1^p + d_2^p)^{1/p}$$

with equality holds when (c_1, c_2) is a constant multiple of (d_1, d_2).

Solution.

(a) Let $f(t) = \alpha t + 1 - \alpha - t^\alpha$. Then, $f'(t) = \alpha\left(1 - t^{\alpha-1}\right)$. So, when $t = 1$, we have $f'(t) = 0$. In fact, $(1, 0)$ is a minimum point (and the only turning point) of $f(t)$. Also, we see that for $t > 1$, f is increasing. As

$$\lim_{t \to 0^+} f(t) = 1 - \alpha > 0,$$

it follows that $f \ge 0$ for all $\alpha \in (0, 1)$ and $t > 0$. As mentioned, equality holds if and only if $t = 1$.

(b) We have

$$\left(\frac{A^p}{B^q}\right)^{1/p} \le \frac{t}{p} + 1 - \frac{1}{p} = \frac{A^p}{pB^q} + \frac{1}{q}.$$

Multiplying both sides by B^q yields

$$A \cdot B^{q - q/p} \le \frac{A^p}{p} + \frac{B^q}{q}.$$

Since $1/p + 1/q = 1$, then $p + q = pq$ so $1 + q/p = q$. The result follows. Equality holds if and only if $t = 1$, i.e. $A^p = B^q$.

(c) From **(b)**, set

$$A = (a_1^p + a_2^p)^{1/p} \text{ and } B = (b_1^q + b_2^q)^{1/q} \quad \text{so} \quad (a_1^p + a_2^p)^{1/p}(b_1^q + b_2^q)^{1/q} \le \frac{a_1^p + a_2^p}{p} + \frac{b_1^q + b_2^q}{q}$$

Let

$$u_i = \frac{a_i}{A} \text{ and } v_i = \frac{b_i}{B} \quad \text{so} \quad u_1^p + u_2^p = 1 \text{ and } v_1^q + v_2^q = 1.$$

By Holder's inequality, we have

$$u_1 v_1 + u_2 v_2 \le \left(u_1^p + u_2^p\right)^{1/p}\left(v_1^q + v_2^q\right)^{1/q} = 1$$

Hence,

$$a_1 b_1 + a_2 b_2 \le AB = (a_1^p + a_2^p)^{1/p}(b_1^q + b_2^q)^{1/q}.$$

The equality case is obvious.

(d) Let $s_1 = c_1 + d_1$ and $s_2 = c_2 + d_2$. Then, apply Minkowski's inequality and we are done. □

Example 4.12 (MA4262 AY24/25 Sem 1 Tutorial 7)**.** Let $(\Omega, \mathscr{A}, \mu)$ be a finite measure space. If f is $(\Omega, \mathscr{A})$ measurable, let $E_n = \{x \in \Omega : n-1 \le |f(x)| < n\}$. Show that $f \in L^1(\Omega, \mathscr{A}, \mu)$ if and only if

$$\sum_{n=1}^{\infty} n\mu(E_n) < +\infty.$$

In general, we have the following result: $f \in L^p(\Omega, \mathscr{A}, \mu)$ for $1 \le p < \infty$, if and only if

$$\sum_{n=1}^{\infty} n^p \mu(E_n) < +\infty.$$

Solution. We first prove the forward direction. Suppose $f \in L^1(\Omega, \mathscr{A}, \mu)$. Then,

$$\int_{\Omega} |f| \; d\mu \text{ is finite} \quad \text{so} \quad \sum_{n=1}^{\infty} \int_{E_n} |f| \; d\mu \text{ is finite.}$$

Since $|f| \ge n-1$, then we can construct the following lower bound for the above sum:

$$\sum_{n=1}^{\infty} \int_{E_n} |f| \; d\mu \ge \sum_{n=1}^{\infty} (n-1)\,\mu(E_n) \quad \text{which is finite.}$$

Note that

$$\sum_{n=1}^{\infty} n\mu(E_n) = \sum_{n=1}^{\infty} (n-1)\,\mu(E_n) + \sum_{n=1}^{\infty} \mu(E_n) \quad \text{where we used } \mu(\Omega) = \sum_{n=1}^{\infty} \mu(E_n).$$

Hence,

$$\sum_{n=1}^{\infty} n\mu(E_n) \quad \text{is also a finite sum.}$$

The proof of the reverse direction is easier. □

4.2. *The Riez-Fischer Theorem*

Definition 4.5 (convergent and Cauchy sequences)**.** A sequence f_n of a measurable function converges to a measurable function $f \in L^p$ if

$$\|f - f_n\|_p \to 0.$$

A sequence f_n is Cauchy in L^p if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|f_m - f_n\|_p < \varepsilon \quad \text{for all} \quad m,n > N.$$

Definition 4.6 (complete space)**.** A vector subspace V of $\mu(\Omega, \mathscr{A}, \mu)$ is complete with respect to $\|\cdot\|_p$ if every f_n in V that is Cauchy in L^p converges to some element in V.

Theorem 4.5 (Riez-Fischer theorem). The space $L^p(\Omega, \mathscr{A}, \mu)$ is complete with respect to the p-norm $\|\cdot\|_p$. Consequently, $\widehat{L}^p(\Omega, \mathscr{A}, \mu)$ is a complete metirc space with respect to the induced p-norm.

For Cauchy sequences f_n, we wish to find its limit. Wishful thinking:

$$f_n \to f \text{ pointwise} \quad \text{in a dominated fashion.}$$

Example 4.13. Set

$$I_1 = [0,1] \quad I_2 = \left[0, \frac{1}{2}\right] \quad I_3 = \left[\frac{1}{2}, 1\right] \quad I_4 = \left[0, \frac{1}{3}\right] \quad I_5 = \left[\frac{1}{3}, \frac{2}{3}\right] \quad I_6 = \left[\frac{2}{3}, 1\right] \quad I_7 = \left[0, \frac{1}{4}\right]$$

and so on. In general, we can enumerate the intervals as follows:

$$[0,1], \left[0, \frac{1}{2}\right], \left[\frac{1}{2}, 1\right], \ldots, \left[0, \frac{1}{m}\right], \left[\frac{1}{m}, \frac{2}{m}\right], \ldots, \left[\frac{m-1}{m}, 1\right], \ldots$$

Set $f_n = \chi_{I_n}$. Then f_n converges in L^p to f, because

$$\|f_n - f\|_p^p = \int |f_n - f|^p \, d\lambda = \int |f_n|^p \, d\lambda \leq \frac{1}{m}$$

when $n \geq 1 + 2 + \cdots + m$. However, f_n does not converge at any point in $[0,1]$.

Definition 4.7 (convergence and Cauchy in measure). A sequence f_n of measurable functions converges in measure to a measurable function f if for all $\varepsilon > 0$, we have

$$\lim_{n \to \infty} \mu(\{x \in \Omega : |f_n - f| \geq \varepsilon\}) = 0.$$

We say f_n is Cauchy in measure if for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n > N$, we have

$$\lim_{m \to \infty} \lim_{n \to \infty} \mu(\{x \in \Omega : |f_m - f_n| \geq \varepsilon\}) = 0.$$

Remark 4.1. In Probability theory, Definition 4.7 is known as convergence in probability.

Example 4.14 (MA4262 AY24/25 Sem 1 Tutorial 8). True or False? You may assume that all functions are integrable.

(a) If a sequence (f_n) converges to f in L^p, then it converges to f almost everywhere.

(b) If a sequence (f_n) converges to f uniformly, then it converges to f in L^p.

(c) If a sequence (f_n) converges to f in measure, then it converges to f in L^p.

(d) Let (f_n) be a Cauchy sequence in measure with a subsequence (f_{n_k}) converges to a function f in measure. Then (f_n) converges to f in measure.

Solution.

(a) False. Consider the typewriter sequence

$$f_n(x) = 1_I \quad \text{where} \quad I = \left[\frac{n-2^k}{2^k}, \frac{n-2^k+1}{2^k}\right].$$

(b) True. Given that $\sup \|f_n - f\| = 0$, it follows that f_n converges to f in L^p (further justification required).

(c) False. Consider the sequence of functions $f_n : [0,1] \to \mathbb{R}$ defined by

$$f_n(x) = \begin{cases} 1, & \text{if } x \in [0, 1/n]; \\ 0, & \text{otherwise.} \end{cases}$$

Here, $f_n \to 0$ in measure because

$$\lim_{n\to\infty} \mu\left(\{x : |f_n(x) - 0| \geq \varepsilon\}\right) = \lim_{n\to\infty} \mu\left(\left[0, \frac{1}{n}\right]\right) = 0.$$

However,

$$\|f_n\|_p = \left(\int_0^{1/n} 1^p \, dx\right)^{1/p} = \left(\frac{1}{n}\right)^{1/p} = \frac{1}{n^{1/p}},$$

which does not converge to zero if we choose a sequence where this doesn't decay fast enough, such as for $p = 1$. Hence, f_n does not converge to zero in $L^p([0,1])$ for $p = 1$.

(d) True. This follows from the fact that a Cauchy sequence in measure that has a convergent subsequence must converge to the same limit in measure. □

Example 4.15 (MA4262 AY24/25 Sem 1 Tutorial 8)**.** Suppose the sequence (f_n) of measurable functions converges almost everywhere to a real-valued measurable function f and φ is continuous on $\mathbb{R}$ to $\mathbb{R}$. Show that the sequence $(\varphi \circ f_n)$ converges almost everywhere to $\varphi \circ f$.

Solution. We wish to show that for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu\left(\{x \in \Omega : |\varphi \circ f_n - \varphi \circ f| \geq \varepsilon\}\right) = 0.$$

Since f_n is a sequence of measurable functions that converges almost everywhere to f, then there exists $E \subseteq \Omega$, where $\mu(E) = 0$, such that $f_n(x) \to f(x)$ for all $x \in \Omega \backslash E$. The continuity of $\varphi : \mathbb{R} \to \mathbb{R}$ implies

$$\varphi \circ f_n \to \varphi \circ f \quad \text{as } n \to \infty.$$

Hence, $\varphi \circ f_n \to \varphi \circ f$ pointwise for $x \in \Omega \backslash E$. This means that for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for every $n \geq N$, we have

$$|\varphi(f_n(x)) - \varphi(f(x))| < \varepsilon \quad \text{for all } x \in \Omega \backslash E.$$

So,

$$\mu\left(\{x \in \Omega : |\varphi(f_n(x)) - \varphi(f(x))| \geq \varepsilon\}\right) \leq \mu(E) = 0$$

and the result follows. □

Example 4.16 (MA4262 AY24/25 Sem 1 Tutorial 8)**.** Let $(\Omega, \mathscr{A}, \mu)$ be a finite measure space. If $f \in M(\Omega, \mathscr{A})$, we define

$$r(f) = \int \frac{|f|}{1+|f|} d\mu.$$

Show that a sequence (f_n) of measurable functions converges in measure to f if and only if $r(f_n - f) \to 0$.

Solution. We first prove the forward direction. Assume that $f_n \to f$ in measure. This means that for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu(\{\omega \in \Omega : |f_n(\omega) - f(\omega)| \geq \varepsilon\}) = 0.$$

Consider the function

$$\frac{|f_n - f|}{1+|f_n - f|}.$$

Notice that

$$0 \leq \frac{|f_n - f|}{1+|f_n - f|} < 1 \quad \text{and} \quad \frac{|f_n - f|}{1+|f_n - f|} \to 0 \text{ pointwise as } n \to \infty.$$

The latter holds since $f_n \to f$ in measure implies that $f_n \to f$ almost everywhere along a subsequence.

For any $\varepsilon > 0$, we can split the integral as follows:

$$r(f_n - f) = \int \frac{|f_n - f|}{1+|f_n - f|} d\mu = \int_{\{|f_n - f| \geq \varepsilon\}} \frac{|f_n - f|}{1+|f_n - f|} d\mu + \int_{\{|f_n - f| < \varepsilon\}} \frac{|f_n - f|}{1+|f_n - f|} d\mu.$$

For the first integral, since

$$\frac{|f_n - f|}{1+|f_n - f|} < 1 \quad \text{then} \quad \int_{\{|f_n - f| \geq \varepsilon\}} \frac{|f_n - f|}{1+|f_n - f|} d\mu \leq \mu(\{|f_n - f| \geq \varepsilon\}),$$

which tends to 0 as $n \to \infty$ by the assumption that $f_n \to f$ in measure.

For the second integral, since

$$0 \leq \frac{|f_n - f|}{1+|f_n - f|} < \varepsilon \quad \text{on} \quad \{|f_n - f| < \varepsilon\},$$

then

$$\int_{\{|f_n - f| < \varepsilon\}} \frac{|f_n - f|}{1+|f_n - f|} d\mu \leq \varepsilon \mu(\Omega).$$

Since $\mu(\Omega)$ is finite, we can make this term arbitrarily small by choosing ε small enough, independently of n. Thus,

$$\lim_{n\to\infty} r(f_n - f) = 0.$$

We now prove the reverse direction. Assume that $r(f_n - f) \to 0$. Given $\varepsilon > 0$, consider the set $A_{n,\varepsilon} = \{\omega \in \Omega : \varepsilon + 1 \geq |f_n(\omega) - f(\omega)| \geq \varepsilon\}$. We have

$$\frac{|f_n - f|}{1+|f_n - f|} \geq \frac{\varepsilon}{2+\varepsilon} \quad \text{on } A_{n,\varepsilon}.$$

Thus,

$$r(f_n - f) = \int \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu \geq \sum_{\varepsilon=0}^{\infty} \int_{A_{n,\varepsilon}} \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu \geq \sum_{\varepsilon=0}^{\infty} \frac{\varepsilon}{2+\varepsilon} \mu(A_{n,\varepsilon}).$$

Since $r(f_n - f) \to 0$, it follows that $\mu(A_{n,\varepsilon}) \to 0$ as $n \to \infty$. This shows that $f_n \to f$ in measure. □

Proposition 4.3. Let f_n be a sequence of measurable functions. If f_n converges uniformly to a measurable function f or doing so in L^p fashion. Then, f_n converges in measure to f. Replacing convergence with Cauchy, the resulting statement also holds.

Theorem 4.6 (Riez)**.** Let f_n be a sequence of measurable functions which is Cauchy in assure. Then, there exists a subsequence f_{n_k} which is Cauchy almost everywhere and convergent almost everywhere.

Moreover, we can arrange g_k which converges in measure to f.

Proof. We will only prove the first result. For each $k > 0$, obtain N_k such that for all $m, n > N_k$, we have

$$\mu\left\{x \in \Omega : |f_m - f_n| > 2^{-k}\right\} < 2^{-k}.$$

We choose a subsequence g_k of f_n as $g_k = f_{N_k}$. In fact, this works. Set

$$E_k = \left\{x \in \mathbb{R} : |g_k - g_{k+1}| > 2^{-k}\right\} \quad \text{so} \quad \mu(E_k) < 2^{-k}.$$

Define

$$F_k = \bigcup_{j=k}^{\infty} E_j \quad \text{so} \quad \mu(F_k) < 2^{-k+1}$$

If $i \geq j \geq k$ and $x \notin F_k$, then

$$\left|g_i - g_j\right| \leq \left|g_{j+1} - g_j\right| + \left|g_{j+2} - g_{j+1}\right|.$$

Take

$$F = \bigcap_{k=1}^{\infty} E_k \quad \text{which is precisely the definition of } \limsup.$$

Then,

$$\mu(F) = \inf \mu(F_k) = 0.$$

We claim that if $x \notin F$ then the sequence g_n is Cauchy as $x \in F_k$ and the previous calculation implies g_n is Cauchy. Define

$$g(x) = \begin{cases} \lim g_n(x) & \text{if } x \notin F; \\ 0 & \text{otherwise.} \end{cases}$$

Then, $g_n \to g$ almost everywhere. □

Corollary 4.4. If f_n is Cauchy in measure, then there exists a measurable f such that f_n converges to f in measure. Moreover, f is unique up to a set of measure 0.

Proof. Take a subsequence g_n as mentioned in the previous theorems. Say $g_n \to f$ pointwise. Then, g_k differs from f only on a small set and f_n differs from f_n only on a small set. □

Proposition 4.4. Suppose $1 \leq p \leq \infty$ and f_n is a sequence of measurable functions in L^p which is Cauchy in L^p. Then, there exists a subsequence g_k and a measurable function h in L^p such that $|g_k| \leq h$.

Definition 4.8 (nearly uniform convergence)**.** Let f_n be a sequence of functions. We say that f_n converges nearly uniformly to f if for every $\delta > 0$, there exists E_δ with $\mu(E_\delta) < \delta$ such that

$$f_n \text{ converges uniformly to } f \quad \text{on } \Omega \backslash E_\delta.$$

Example 4.17. An example of nearly uniform convergence is the sequence of functions $f_n(x) = x^n$ defined on the interval $[0,1]$. We shall analyse the convergence of $f_n \to f$, where

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1; \\ 1 & \text{if } x = 1. \end{cases}$$

For every $\delta > 0$, define $E_\delta = [1-\delta/2, 1]$, and we see that $\mu(E_\delta) = \delta/2 < \delta$. On the set $[0,1] \setminus E_\delta = [0, 1-\delta/2)$, the sequence $f_n(x) = x^n$ converges uniformly to 0. This is because for $x \in [0, 1-\delta/2)$, we have $0 \leq x < 1$, so as $n \to \infty$, $x^n \to 0$ uniformly on this set.

On the set $E_\delta = [1-\delta/2, 1]$, the convergence is not uniform, since $f_n(x) = x^n$ approaches 1 as $x \to 1$, and it becomes more erratic near $x = 1$. Hence $f_n(x) = x^n$ converges nearly uniformly to $f(x)$, with the set E_δ (where uniform convergence fails) having measure less than δ.

Lemma 4.1. If f_n converges in L^∞ to f, then f_n converges nearly uniformly to f.

Proof. Recall that convergence of f in L^∞ holds if and only if f converges uniformly up to a set of measure 0. The result follows. □

Lemma 4.2. If f_n converges nearly uniformly to f, then f_n converges in measure to f.

Theorem 4.7. Suppose $\mu(\Omega) < \infty$ and f_n is a sequence of measurable real-valued functions which converges almost everywhere on Ω to a measurable real-valued function f. Then,

$$f_n \quad \text{converges nearly uniformly and in measure} \quad \text{to } f.$$

5. Application to Probability

For this section, let $(\Omega, \mathscr{F}, P)$ be a probability space with $P(\Omega) = 1$. In other words, $(\Omega, \mathscr{F}, P)$ is a measure space with $P(\Omega) = 1$.

Definition 5.1 (random variable, expectation and variance)**.** A random variable X is a measurable function from $\mathbb{R}$ to $\mathbb{R}$. The expectation of such X is just

$$\mathbb{E}[X] = \int X \, d\mu.$$

The variance of X is given by

$$\int (X - \mathbb{E}[X])^2 \, d\mu = \|X - \mathbb{E}[X]\|_2^2.$$

Definition 5.2 (covariance)**.** For random variables X and Y, their covariance $\operatorname{cov}(X,Y)$ is denoted by

$$\int_\Omega (X - \mathbb{E}[X])(Y - \mathbb{E}[Y]) \, d\mu = \langle X - \mathbb{E}[X], Y - \mathbb{E}[Y] \rangle$$

where we recall that

$$\langle f, g \rangle = \int fg \, d\mu = \int \frac{(f+g)^2 - f^2 - g^2}{2} \, d\mu = \frac{\|f+g\|_2^2 - \|f\|_2^2 - \|g\|_2^2}{2}.$$

Definition 5.3 (uncorrelated random variables)**.** A sequence X_n of random variables is uncorrelated if $\operatorname{cov}(X_n, X_m) = 0$ for all $m \neq n$.

Definition 5.4 (covariance stationary)**.** A sequence of random variables X_n is covariance stationary if

$$\mathbb{E}[X_n] \text{ is constant} \quad \text{and} \quad \operatorname{cov}(X_n, X_{n+k}) \text{ is constant for all } k \geq 0.$$

In particular, $\operatorname{Var}(X_n) = \operatorname{cov}(X_n, X_n)$ is a constant.

Theorem 5.1 (Chebyshev's weak law of large numbers)**.** Suppose X_n is an uncorrelated and covariance stationary sequence of random variables. Define

$$\overline{X_n} = \frac{X_1 + \ldots + X_n}{n}.$$

Then, $\overline{X_n}$ converges in measure/probability to $\mathbb{E}[X_n]$ for any positive integer n.

Proof. We will show a stronger statement, i.e. that $\overline{X_n}$ converges in L^2 to μ. Note that $X_n \in L^2$ since $\text{Var}(X_n)$ is finite. Then,

$$\int \left(\overline{X_n} - \alpha\right)^2 \, d\mu = \int \left(\frac{X_1 + \ldots + X_n}{n} - \mu\right)^2 \, d\mu = \int \left[\frac{(X_1 - \mu) + \ldots + (X_n - \alpha)}{n}\right]^2 \, d\mu$$

which is equal to

$$\begin{aligned}
\frac{1}{n^2} \sum_{1 \leq i,j \leq n} \int (X_i - \mu)(X_j - \mu) \, d\mu &= \frac{1}{n^2} \sum_{i=1}^{n} \int (X_i - \mu)^2 \, d\mu \quad \text{by applying uncorrelated property} \\
&= \frac{1}{n^2} \sum_{i=1}^{n} \text{cov}(X_i, X_i) \\
&= \frac{1}{n^2} \sum_{i=1}^{n} \text{Var}(X_i) \\
&= \frac{1}{n^2} \cdot n \, \text{Var}(X_1) \quad \text{since } \text{Var}(X_1) = \ldots = \text{Var}(X_n) \\
&= \frac{\text{Var}(X_1)}{n} \quad \text{which tends to } 0
\end{aligned}$$

As we have shown that the integral of $\left(\overline{X_n} - \mu\right)^2$ goes to 0, it shows that $\overline{X_n}$ converges in measure to $\mathbb{E}[X_n]$. □

Definition 5.5 (independent events and independent random variables)**.** Let X be a random variable with distribution function F_X. Then,

$$F_X(\alpha) = \mu\{\omega \in \Omega : X(\omega) \geq \alpha\}.$$

Two events $X, Y \in \Omega$ (measurable subsets of Ω are independent if

$$\mu(X \cap Y) = \mu(X)\mu(Y).$$

Two random variables X and Y are independent if

$$\text{for all } \alpha, \beta, \quad E_\alpha = \{\omega \in \Omega : X(\omega) \geq \alpha\}, F_\beta = \{\omega \in \Omega : X(\omega) \geq \beta\} \quad \text{are independent events.}$$

Proposition 5.1. If X and Y are independent random variables, then

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y].$$

Proof. For the special case where $X = \chi_A$ and $Y = \chi_B$, we wish to ask whether

$$\int \chi_A \cdot \chi_B \, d\mu = \int \chi_A \, d\mu \int \chi_B \, d\mu.$$

We note that

$$\int \chi_A \cdot \chi_B \, d\mu = \int \chi_{A \cap B} \, d\mu = \mu(A \cap B) = \mu(A)\mu(B)$$

and the result follows. □

Corollary 5.1. If X and Y are independent, then they are uncorrelated, i.e. $\mathrm{cov}(X,Y) = 0$.

Proof. We have

$$\begin{aligned}\mathrm{cov}(X,Y) &= \int (X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\,d\mu \\ &= \int XY\,d\mu - \mathbb{E}[X]\int Y\,d\mu - \mathbb{E}[Y]\int X\,d\mu + \mathbb{E}[X]\mathbb{E}[Y] \\ &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] - \mathbb{E}[Y]\mathbb{E}[X] + \mathbb{E}[X]\mathbb{E}[Y] \\ &= 0 \quad \text{since } \mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] \text{ by Proposition 5.1}\end{aligned}$$

so it follows that $\mathrm{cov}(X,Y) = 0$. □

Definition 5.6 (independent and identically distributed random variables)**.** A sequence X_n of random variables is identically distributed if

$$F_{X_i} = F_{X_j} \quad \text{for all } i, j \in \mathbb{N}.$$

The common abbreviation is i.i.d. RV.

Corollary 5.2. Suppose X_n is a sequence of i.i.d. RV. Assume that X_n has finite mean and variance. Take

$$\overline{X_n} = \frac{X_1 + \ldots + X_n}{n}.$$

Then, $\overline{X_n}$ converges in measure to measure to μ, where

$$\mu = \mathbb{E}[X_1] = \int X\,d\mu.$$

Proof. Note that $\mathrm{cov}(X_n, X_m) = 0$ for distinct m and n by Corollary 5.1. Hence, $\mathrm{cov}(X_n, X_m) = 0$, which implies X_n and X_m are uncorrelated. Also, $\mathbb{E}[X_n] = P(X_n \geq -\infty)$. Hence, it follows that $\mathbb{E}[X_n]$ is a constant μ as n varies. For $\alpha \geq 0$, the random variable $Y_n = (X_n - \mu)^2$ is such that

$$P(Y_n \geq \alpha) = P(X_n \geq \sqrt{\alpha} + \mu) + P(X_n \leq -\sqrt{\alpha} + \mu).$$

In particular, it gives us that $\mathbb{E}[Y_n]$ is a constant as n varies. Hence, $\mathrm{Var}(X_n) = \mathbb{E}[Y_n]$ is a constant as n varies, implying that X_n is covariance stationary (Definition 5.4). The desired conclusion follows from Chebyshev's weak law of large numbers (Theorem 5.1). □

Proposition 5.2 (Chebysehv's inequality)**.** Suppose X is a random variable with mean α and variance σ^2. Then, for all $\gamma > 0$, we have

$$\mu\{\omega \in \Omega : |X(\omega) - \alpha| > \gamma\sigma\} \leq \frac{1}{\gamma^2}.$$

Proof. We have

$$\sigma^2 = \int_\Omega (X-\alpha)^2 \, d\mu \geq \int_{|X-\alpha|\geq\gamma\sigma} (X-\alpha)^2 \, d\mu \geq \int_{|X-\alpha|\geq\gamma\sigma} \gamma^2\sigma^2 \, d\mu \geq \gamma^2\sigma^2\mu\,\{\omega\in\Omega : |X(\omega)-\alpha| > \gamma\sigma\}$$

The result follows immediately. □

Theorem 5.2 (Kolomogorov's strong law of large numbers)**.** Suppose X_n is an i.i.d. sequence of random variables with finite mean and variance. Setting

$$\overline{X_n} = \frac{X_1 + \ldots + X_n}{n},$$

the sequence $\overline{X_n}$ converges almost everywhere almost surely to μ, where

$$\mu = \mathbb{E}[X_1] = \int X \, d\mu.$$

6. Differentiation

6.1. *Differentiation of Monotone Functions*

Suppose $f(x):[a,b]\to\mathbb{R}$ is integrable. Suppose we define the antiderivative F as follows:

$$F(x)=\int_a^x f(x)\,d\mu.$$

Do we know that $F'(x)=f(x)$ or $F'(x)=f(x)$almost everywhere?

Definition 6.1 (Vitali cover). A collection Γ of intervals I is a Vitali cover of a set E if for every $\varepsilon>0$ and $x\in E$,

$$I\subseteq\Gamma \quad \text{such that} \quad x\in I \text{ and } \ell(I)<\varepsilon.$$

Theorem 6.1 (Vitali covering theorem). Let $E\subseteq\mathbb{R}$ be measurable such that $\lambda^*(E)<\infty$ (recall that λ^* is the outer Lebesgue measure) and Γ is a Vitali cover. Then, for every $\varepsilon>0$, there exists a finite collection of disjoint intervals $\{I_1,\ldots,I_n\}\subseteq\Gamma$ such that

$$\lambda^*(E\backslash S)<\varepsilon \quad \text{where} \quad S=\bigcup_{j=1}^n I_j.$$

One can see the Vitali covering theorem (Theorem 6.1) as a refinement of inner regularity.

Definition 6.2 (Dini derivative). Let f be a real-valued function defined in a neighborhood of x_0. Define the Dini derivatives $D^+f(x_0)$, $D^-f(x_0)$, $D_+f(x_0)$, and $D_-f(x_0)$ as follows:

(1) Upper-right Dini derivative:

$$D^+f(x_0)=\limsup_{h\to0^+}\frac{f(x_0+h)-f(x_0)}{h}$$

(2) Lower-right Dini derivative:

$$D_+f(x_0)=\liminf_{h\to0^+}\frac{f(x_0+h)-f(x_0)}{h}$$

(3) Upper-left Dini derivative:

$$D^-f(x_0)=\limsup_{h\to0^-}\frac{f(x_0+h)-f(x_0)}{h}$$

(4) Lower-left Dini derivative:

$$D_-f(x_0)=\liminf_{h\to0^-}\frac{f(x_0+h)-f(x_0)}{h}$$

The Dini derivatives always exist but it is possible for them to be $\pm\infty$.

Theorem 6.2. Let f be a non-increasing real-valued function on $[a,b]$. Then, f is differentiable almost everywhere. The derivative $f'(x)$ is measurable almost everywhere too. Moreover,

$$\int_a^b f'(x)\,dx \leq f(b)-f(a).$$

The proof is quite long. We will only deal with the second half of the proof. For $x\in[0,a]$, define

$$g(x)=\begin{cases}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} & \text{if } D^+=D^-=D_+=D_-;\\ 0 & \text{otherwise.}\end{cases}$$

Note that $g(x):[a,b]\to\mathbb{R}\cup\{\pm\infty\}$. Set

$$g_n(x)=\frac{f(x+1/n)-f(x)}{x}\quad\text{where}\quad f(z)=f(b)\text{ if } z>b.$$

Then, $g_n\to g$ almost everywhere. As f is non-decreasing, then $g_n(x)\geq 0$. As such, $g(x)\geq 0$ almost everywhere. To get $g(x)$ only ∞ on a set of measure zero, it suffices to show that

$$\int g(x)\,d\mu<\infty\quad\text{but a stricter inequality is}\quad\int g(x)\,d\mu<f(b)-f(a).$$

By Fatou's lemma, we have

$$\begin{aligned}\int g(x)\,d\mu &\leq \liminf_{n\to\infty}\int g_n(x)\,d\mu\\ &=\liminf_{n\to\infty}\int\frac{f(x+1/n)-f(x)}{1/n}\,d\mu\\ &=\liminf_{n\to\infty} n\left[\int f\left(x+\frac{1}{n}\right)d\mu-\int f(x)\,d\mu\right]\\ &=\liminf_{n\to\infty} n\left[\int_{[b,b+1/n]} f(x)\,d\mu-\int_{[a,a+1/n]} f(x)\,d\mu\right]\\ &\leq f(b)-f(a)\quad\text{but this needs further justification}\end{aligned}$$

6.2. *Functions with Bounded Variation*

If $r\in\mathbb{R}$, let

$$r^+=\max\{r,0\}\text{ and } r^-=\max\{-r,0\}\quad\text{so}\quad r=r^+-r^-.$$

Definition 6.3 (bounded variation). Let $f:[a,b]\to\mathbb{R}$ and $\Delta=\{a=x_0<x_1<\ldots<x_n=b\}$ be a partition. Then, define

$$p_\Delta=\sum_{i=1}^n\left(f(x_i)-f(x_{i-1})\right)^+\quad\text{and}\quad n_\Delta=\sum_{i=1}^n\left(f(x_i)-f(x_{i-1})\right)^-.$$

Set $t_\Delta = p_\Delta + n_\Delta$. We also define P_a^b, N_a^b, T_a^b to be the following:

$$P_a^b = \sup_\Delta p_\Delta \quad \text{and} \quad N_a^b = \sup n_\Delta \quad \text{and} \quad T_a^b = P_a^b + N_a^b = \sup_\Delta t_\Delta.$$

If f has $T_a^b < \infty$, we say that f has bounded variation over $[a,b]$. The space of all such functions is denoted by $\text{BV}[a,b]$.

Theorem 6.3 (Jordan decomposition theorem)**.** A function of bounded variation is the difference between two monotone, non-decreasing real-valued functions.

Corollary 6.1. If f has bounded variation, then f' is well-defined almost everywhere.

6.3. *The Fundamental Theorem of Calculus*

Lemma 6.1. If f is integrable on $[a,b]$, then

$$F(x) = \int_a^x f(t)\, dt = \int f \cdot \chi_{[a,x]}\, d\mu$$

is a continuous function with bounded variation.

Proof. Suppose $x_n \to x$. Then, we are interested in whether

$$F(x_n) \to F(x) \quad \text{or equivalently} \quad \int f \cdot \chi_{[a,x_n]}\, d\mu \to \int f \cdot \chi_{[a,x]}\, d\mu.$$

Note that

$$\chi_{[a,x_n]} \to \chi_{[a,x]} \text{ almost everywhere} \quad \text{implies} \quad f \cdot \chi_{[a,x_n]} \to f \cdot \chi_{[a,x]} \text{ almost everywhere.}$$

Also, note that

$$\left| f \cdot \chi_{[a,x_n]} \right| \le |f|.$$

By the Lebesgue dominated convergence theorem,

$$\int f \cdot \chi_{[a,x_n]}\, d\mu \to \int f \cdot \chi_{[a,x]}\, d\mu.$$

We wish to show that F has bounded variation over $[a,b]$, i.e. $T_a^b < \infty$. We will take $\Delta = \{x_0, x_1, \ldots, x_n\}$ to be a partition of $[a,b]$, where $x_0 = a$ and $x_n = b$. Then,

$$t_\Delta = \sum_{i=1}^n |F(x_i) - F(x_{i-1})| = \sum_{i=1}^n \left| \int_{x_{i-1}}^{x_i} f(t)\, dt \right| \le \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f(t)|\, dt = \int_a^b |f(t)|\, dt$$

which is finite as f is integrable. □

Lemma 6.2 (absolute continuity of integral)**.** Let f be integrable on a set E. Then, for $\varepsilon > 0$, there exists $\delta > 0$ such that if $A \subseteq E$ with $\mu(A) < \delta$, then

$$\int_A f \, d\mu < \varepsilon.$$

Proof. Let $f = f^+ - f^-$. It suffices to show the same statement for f^+ and f^-. Without loss of generality, suppose $f \geq 0$. Then, set

$$f_n(x) = \begin{cases} f(x) & \text{if } f(x) \leq n; \\ n & \text{otherwise.} \end{cases}$$

Then, $f_n \to f$ pointwise. By the monotone convergence theorem,

$$\int_E f_n \, d\mu \to \int_E f \, d\mu.$$

Suppose for n large enough, we have

$$\int_E (f - f_n) \, d\mu < \frac{\varepsilon}{2}.$$

Choose $\delta < \varepsilon/2n$. Then,

$$\int_A f \, d\mu = \int_A f_n \, d\mu + \int_A f - f_n \, d\mu < \frac{\varepsilon}{2n} \cdot n + \frac{\varepsilon}{2} = \varepsilon$$

and the result follows. □

Lemma 6.3. If f is integrable on $[a,b]$ and

$$\int_a^x f(t) \, dt = 0 \quad \text{for all } x \in [a,b],$$

then $f(t) = 0$ almost everywhere.

Proof. We shall prove this by contradiction. Set E_n to be the set of all functions $f(x)$ such that $f(x) > 1/n$. Then,

$$E = \bigcup_{n \in \mathbb{N}} E_n = \{f(x) : f(x) > 0\} \quad \text{has} \quad \mu(E) > 0.$$

So, $\mu(E_n) > 0$ for some $n \in \mathbb{N}$. We can choose some closed subset F of E_n of positive measure. Take $S = (a,b) \backslash F$. Then,

$$\int_a^b f(t) \, dt = 0 \quad \text{and} \quad \int_F f(t) \, dt > 0.$$

Hence,

$$\int_S f \, d\mu < 0.$$

Recall that S is a countable disjoint union of open intervals. One must have, by countable additivity, that

$$\int f\, d\mu \neq 0.$$

Result follows easily from here. □

Lemma 6.4. If f is a measurable function bounded on $[a,b]$, and

$$F(x) = \int_a^x f(t)\, dt + F(a),$$

then $F'(x) = f(x)$ for $x \in [a,b]$ almost everywhere.

Proof. Note that $F(x)$ is a bounded variation on $[a,b]$, so $F'(x)$ exists almost everywhere. Then, set

$$f_n(x) = \frac{F(x+1/n) - F(x)}{1/n} = n \int_x^{x+1/n} f(t)\, dt.$$

We already know that $f_n \to F'(x)$ almost everywhere by definition, so we only need to check that $f_n \to f$ almost everywhere. Suppose $f(x)$ is bounded by some constant K, i.e. there exists K such that $|f_n(x)| \leq K$. Hence,

$$\begin{aligned}
\int_a^c F'(x)\, dx &= \lim_{n\to\infty} \int_a^c f_n(x)\, dx \quad \text{by dominated convergence theorem} \\
&= \lim_{h\to 0} \frac{1}{h} \int_a^c F(x+h) - F(x)\, dx \quad \text{by setting } h = \frac{1}{n} \\
&= \lim_{h\to 0} \frac{1}{h} \left(\int_a^c F(x+h)\, dx - \int_a^c F(x)\, dx \right) \\
&= \lim_{h\to 0} \frac{1}{h} \left(\int_{a+h}^{c+h} F(x)\, dx - \int_a^c F(x)\, dx \right) \\
&= \lim_{h\to 0} \frac{1}{h} \left(\int_c^{c+h} F(x)\, dx - \int_a^{a+h} F(x)\, dx \right) \\
&= F(c) - F(a) \quad \text{since } F \text{ is constant} \\
&= F(c) - F(a) \\
&= \int_a^c f\, d\mu
\end{aligned}$$

As such,

$$\int_a^c F'(x) - f\, d\mu = 0,$$

and we conclude that $F'(x) = f(x)$ for $x \in [a,b]$ almost everywhere. □

Theorem 6.4 (fundamental theorem of Calculus). Let $f : [a,b] \to \mathbb{R}$ be integrable and

$$F(x) = \int_a^x f(t)\, dt + F(a).$$

Then, $F'(x) = f(x)$ almost everywhere.

6.4. *Absolute Continuity*

Definition 6.4. A function $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{i\in I} \left| f\left(x_i'\right) - f\left(x_i\right) \right| < \varepsilon.$$

Here, $\{(x_i, x_i')\}_{i\in I}$ is a collection of disjoint intervals such that

$$\sum_{i\in I} \left| x_i - x_i' \right| < \delta.$$

Theorem 6.5. A function is an indefinite integral if and only if it is absolutely continuous.

Lemma 6.5. If f is absolutely continuous, then f is of bounded variation.

Proof. Suppose f is absolutely continuous, then for each ε, there exists $\delta > 0$ such that

$$\text{the intervals} \quad [x_1, y_1], \ldots, [x_n, y_n] \quad \text{are disjoint.}$$

Note that each $[x_i, y_i] \subseteq [a,b]$. The mentioned intervals are disjoint with

$$\sum_{i=1}^{n} |y_i - x_i| < \delta \quad \text{which implies} \quad \sum_{i=1}^{n} |f(y_i) - f(x_i)| < \varepsilon.$$

Note that f is a bounded variation if and only if there exists M such that for every partition $\{x_0, x_1, \ldots, x_n\}$, we have

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \le M.$$

Take $\varepsilon = 1$ and obtain a corresponding δ. Take $\Delta_0 = \{y_0, \ldots, y_n\}$ to be a partition of $[a,b]$, where $y_0 = a$ and $y_n = b$, such that $|y_i - y_{i-1}| < \delta$. Now, take an arbitrary partition of $[a,b]$, say $\Delta = \{x_0, x_1, \ldots, x_n\}$. We claim that

$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| < \frac{2}{\delta}.$$

Without loss of generality, Δ refines Δ_0, so the result follows. □

Lemma 6.6. If f is absolutely continuous and $f' = 0$ almost everywhere, then $f = c$ almost everywhere.

6.5. *Lebesgue Differentiation Theorem and the Radon-Nikodym Theorem*

Say we wish to generalise the FTC to higher dimensions like $\mathbb{R}^d$. What do we do? Let

$$f : \prod_{i=1}^{d} (a_i, b_i) \to \mathbb{R} \quad \text{be integrable.}$$

Here, let Ω denote the mentioned Cartesian product, i.e. the domain. For each measurable subset $A \subseteq \Omega$, this defines a map

$$A \mapsto \int f \, d\mu.$$

For each $x \in \Omega$, define the derivative of this integral operator at x to be

$$\lim_{\varepsilon \to 0} \frac{1}{\mu(B(\mathbf{x}, \varepsilon))} \int_{B(\mathbf{x},\varepsilon)} f \, d\mu.$$

Theorem 6.6 (Lebesgue differentiation theorem)**.** For almost every $x \in \Omega$, we have

$$f(x) = \lim_{\varepsilon \to 0} \frac{1}{\mu(B(\mathbf{x}, \varepsilon))} \int_{B(\mathbf{x},\varepsilon)} f \, d\mu.$$

In the proof of this result, we use the Vitali covering lemma to construct an L^1 bound of the Hardy-Littlewood maximal function.

We have another question. What is the generalisation of the absolute continuity characterisation of integrals? The answer involves the Radon-Nikodym theorem. In our course, measure is merely a way to assign volume. From a different perspective, measure is a generalisation of a function (Geometric Measure Theory).

Suppose we have a non-negative measurable function $f : \Omega \to \mathbb{R}$. One can define a measure μ_f as follows:

$$\mu_f(A) = \int_A f \, d\mu.$$

Definition 6.5 (absolute continuity)**.** Let μ, ν be measures on Ω. We say that

$$\nu \text{ is absolutely continuous} \quad \text{with respect to} \quad \mu$$

if

$$\text{for all measurable } E \subseteq \Omega \quad \text{we have} \quad \mu(E) = 0 \text{ implies } \nu(E) = 0.$$

Lemma 6.7. Let f be measurable from Ω to $\mathbb{R}$. Then, μ_f is absolutely continuous with respect to μ.

Lemma 6.8 (finite and σ-finite measure)**.** A measure μ is finite if it takes values in $\mathbb{R}$ for all $E \subseteq \Omega$; a measure ν is σ-finite if it is a countable sum of finite measure.

Theorem 6.7 (Radon-Nikodym)**.** Suppose μ and ν are σ-finite measures on Ω and ν is absolutely continuous with respect to μ. Then, there exists a measurable function f with respect to μ such that

$$\nu(A) = \int_A f \, d\mu.$$